Proposition 4.2.11. *Let X be a t.v.s. and p a seminorm on X. Then the following conditions are equivalent:*
a) the open unit semiball $\mathring{U}_p$ of p is an open set.
b) p is continuous at the origin.
c) the closed unit semiball U_p of p is a barrelled neighbourhood of the origin.
d) p is continuous at every point.

Proof.
a) $\Rightarrow$ *b)* Suppose that $\mathring{U}_p$ is open in the topology on X. Then for any $\varepsilon > 0$ we have that $p^{-1}([0,\varepsilon[) = \{x \in X : p(x) < \varepsilon\} = \varepsilon\mathring{U}_p$ is an open neighbourhood of the origin in X. This is enough to conclude that $p : X \to \mathbb{R}^+$ is continuous at the origin.

b) $\Rightarrow$ *c)* Suppose that p is continuous at the origin, then $U_p = p^{-1}([0,1])$ is a closed neighbourhood of the origin. Since U_p is also absorbing and absolutely convex by Proposition 4.2.10-a), U_p is a barrel.

c) $\Rightarrow$ *d)* Assume that c) holds and fix $o \neq x \in X$. Using Proposition 4.2.10 and Proposition 4.2.3, we get that for any $\varepsilon > 0$: $p^{-1}([-\varepsilon + p(x), p(x) + \varepsilon]) = \{y \in X : |p(y) - p(x)| \leq \varepsilon\} \supseteq \{y \in X : p(y - x) \leq \varepsilon\} = x + \varepsilon U_p$, which is a closed neighbourhood of x since X is a t.v.s. and by the assumption *c*). Hence, p is continuous at x.

d) $\Rightarrow$ *a)* If p is continuous on X then a) holds because the preimage of an open set under a continuous function is open and $\mathring{U}_p = p^{-1}([0,1[)$. □

With such properties in our hands we are able to give a criterion to compare two locally convex topologies on the same space using their generating families of seminorms.

Theorem 4.2.12 (Comparison of l.c. topologies)**.**
Let $\mathcal{P} = \{p_i\}_{i\in I}$ and $\mathcal{Q} = \{q_j\}_{j\in J}$ be two families of seminorms on the vector space X inducing respectively the topologies $\tau_\mathcal{P}$ and $\tau_\mathcal{Q}$, which both make X into a locally convex t.v.s.. Then $\tau_\mathcal{P}$ is finer than $\tau_\mathcal{Q}$ (i.e. $\tau_\mathcal{Q} \subseteq \tau_\mathcal{P}$) iff

$$\forall q \in \mathcal{Q}\ \exists n \in \mathbb{N},\, i_1, \ldots, i_n \in I,\, C > 0 \ \ s.t.\ \ Cq(x) \leq \max_{k=1,\ldots,n} p_{i_k}(x),\ \forall x \in X. \tag{4.2}$$

Proof.
Let us first recall that, by Theorem 4.2.9, we have that

$$\mathcal{B}_\mathcal{P} := \left\{ \bigcap_{k=1}^{n} \varepsilon\mathring{U}_{p_{i_k}} : i_1, \ldots, i_n \in I, n \in \mathbb{N}, \varepsilon > 0, \varepsilon \in \mathbb{R} \right\}$$

and

$$\mathcal{B}_{\mathcal{Q}} := \left\{ \bigcap_{k=1}^{n} \varepsilon \mathring{U}_{q_{j_k}} : j_1, \ldots, j_n \in J, n \in \mathbb{N}, \varepsilon > 0, \varepsilon \in \mathbb{R} \right\}.$$

are respectively bases of neighbourhoods of the origin for $\tau_{\mathcal{P}}$ and $\tau_{\mathcal{Q}}$.

By using Proposition 4.2.10, the condition (4.2) can be rewritten as

$$\forall q \in \mathcal{Q},\ \exists n \in \mathbb{N},\ i_1, \ldots, i_n \in I,\ C > 0 \text{ s.t. } C \bigcap_{k=1}^{n} \mathring{U}_{p_{i_k}} \subseteq \mathring{U}_q.$$

which means that

$$\forall q \in \mathcal{Q}, \exists\ B_q \in \mathcal{B}_{\mathcal{P}} \text{ s.t. } B_q \subseteq \mathring{U}_q. \tag{4.3}$$

since $C \bigcap_{k=1}^{n} \mathring{U}_{p_{i_k}} \in \mathcal{B}_{\mathcal{P}}$.

Condition (4.3) means that for any $q \in \mathcal{Q}$ the set $\mathring{U}_q \in \tau_{\mathcal{P}}$, which by Proposition 4.2.11 is equivalent to say that q is continuous w.r.t. $\tau_{\mathcal{P}}$. By definition of $\tau_{\mathcal{Q}}$, this gives that $\tau_{\mathcal{Q}} \subseteq \tau_{\mathcal{P}}$. [1]

□

This theorem allows us to easily see that the topology induced by a family of seminorms on a vector space does not change if we close the family under taking the maximum of finitely many of its elements. Indeed, the following result holds.

Proposition 4.2.13. *Let $\mathcal{P} := \{p_i\}_{i\in I}$ be a family of seminorms on a vector space X and $\mathcal{Q} := \left\{ \max_{i\in B} p_i : \emptyset \neq B \subseteq I \text{ with } B \text{ finite} \right\}$. Then $\mathcal{Q}$ is a family of seminorms and $\tau_{\mathcal{P}} = \tau_{\mathcal{Q}}$, where $\tau_{\mathcal{P}}$ and $\tau_{\mathcal{Q}}$ denote the topology induced on X by $\mathcal{P}$ and $\mathcal{Q}$, respectively.*

Proof.
First of all let us note that, by Proposition 4.2.10, $\mathcal{Q}$ is a family of seminorms. On the one hand, since $\mathcal{P} \subseteq \mathcal{Q}$, by definition of induced topology we have $\tau_{\mathcal{P}} \subseteq \tau_{\mathcal{Q}}$. On the other hand, for any $q \in \mathcal{Q}$ we have $q = \max_{i\in B} p_i$ for some $\emptyset \neq B \subseteq I$ finite. Then (4.2) is fulfilled for $n = |B|$ (where $|B|$ denotes the cardinality of the finite set B), $i_1, \ldots, i_n$ being the n elements of B and for any $0 < C \leq 1$. Hence, by Theorem 4.2.12, $\tau_{\mathcal{Q}} \subseteq \tau_{\mathcal{P}}$. □

[1] **Alternate proof without using Prop 4.2.11** (Exercise Sheet 5).

This fact can be used to show the following very useful property of locally convex t.v.s.

Proposition 4.2.14. *The topology of a locally convex t.v.s. can be always induced by a* directed *family of seminorms.*

Definition 4.2.15. *A family $\mathcal{Q} := \{q_j\}_{j\in J}$ of seminorms on a vector space X is said to be* directed *if*

$$\forall\, j_1, j_2 \in J,\ \exists\, j \in J, C > 0 \ \ s.t.\ \ Cq_j(x) \geq \max\{q_{j_1}(x), q_{j_2}(x)\},\ \forall\, x \in X \quad (4.4)$$

or equivalently by induction if

$$\forall\, n \in \mathbb{N}, j_1, \ldots, j_n \in J,\ \exists\, j \in J, C > 0 \ \ s.t.\ \ Cq_j(x) \geq \max_{k=1,\ldots,n} q_{j_k}(x),\ \forall\, x \in X.$$

Proof. of Proposition 4.2.14
Let (X, τ) be a locally convex t.v.s.. By Theorem 4.2.9, we have that there exists a family of seminorms $\mathcal{P} := \{p_i\}_{i\in I}$ on X s.t. $\tau = \tau_{\mathcal{P}}$. Let us define $\mathcal{Q}$ as the collection obtained by forming the maximum of finitely many elements of $\mathcal{P}$, i.e. $\mathcal{Q} := \left\{ \max_{i\in B} p_i : \emptyset \neq B \subseteq I \text{ with } B \text{ finite} \right\}$. By Proposition 4.2.13, $\mathcal{Q}$ is a family of seminorms and we have that $\tau_{\mathcal{P}} = \tau_{\mathcal{Q}}$.

Let $q, q' \in \mathcal{Q}$, i.e. $q := \max_{i\in B} p_i$ and $q' := \max_{i\in B'} p_i$ for some non-empty finite subsets B, B' of I. Let us define $q'' := \max_{i\in B\cup B'} p_i$. Then $q'' \in \mathcal{Q}$ and for any $C \geq 1$ we have that (4.4) is satisfied, because we get that for any $x \in X$

$$Cq''(x) = C \max\left\{ \max_{i\in B} p_i(x), \max_{i\in B'} p_i(x) \right\} \geq \max\{q(x), q'(x)\}.$$

Hence, $\mathcal{Q}$ is directed. □

It is possible to show (Exercise Sheet 5) that a basis of neighbourhoods of the origin for the l.c. topology $\tau_{\mathcal{Q}}$ induced by a directed family of seminorms $\mathcal{Q}$ is given by:

$$\mathcal{B}_d := \{r\mathring{U}_q : q \in \mathcal{Q}, r > 0\}. \quad (4.5)$$

4.3 Hausdorff locally convex t.v.s

In Section 2.2, we gave some characterization of Hausdorff t.v.s. which can of course be applied to establish whether a locally convex t.v.s. is Hausdorff or not. However, in this section we aim to provide necessary and sufficient conditions bearing only on the family of seminorms generating a locally convex topology for being a Hausdorff topology.

Definition 4.3.1.
A family of seminorms $\mathcal{P} := \{p_i\}_{i\in I}$ *on a vector space* X *is said to be* separating *if*

$$\forall x \in X \setminus \{o\}, \exists\ i \in I \ s.t.\ p_i(x) \neq 0. \tag{4.6}$$

Note that the separation condition (4.6) is equivalent to

$$p_i(x) = 0,\ \forall i \in I \ \Rightarrow x = o$$

which by using Proposition 4.2.10 can be rewritten as

$$\bigcap_{i\in I, c>0} c\mathring{U}_{p_i} = \{o\},$$

since $p_i(x) = 0$ is equivalent to say that $p_i(x) < c$, for all $c > 0$.

It is clear that if any of the elements in a family of seminorms is actually a norm, then the the family is separating.

Lemma 4.3.2. *Let* $\tau_{\mathcal{P}}$ *be the topology induced by a separating family of seminorms* $\mathcal{P} := (p_i)_{i\in I}$ *on a vector space* X*. Then* $\tau_{\mathcal{P}}$ *is a Hausdorff topology.*

Proof. Let $x, y \in X$ be such that $x \neq y$. Since $\mathcal{P}$ is separating, we have that $\exists\ i \in I$ with $p_i(x - y) \neq 0$. Then $\exists\ \epsilon > 0$ s.t. $p_i(x - y) = 2\epsilon$. Let us define $V_x := \{u \in X \mid p_i(x - u) < \epsilon\}$ and $V_y := \{u \in X \mid p_i(y - u) < \epsilon\}$. By Proposition 4.2.10, we get that $V_x = x + \varepsilon\mathring{U}_{p_i}$ and $V_y = y + \varepsilon\mathring{U}_{p_i}$. Since Theorem 4.2.9 guarantees that $(X, \tau_{\mathcal{P}})$ is a t.v.s. where the set $\varepsilon\mathring{U}_{p_i}$ is a neighbourhood of the origin, V_x and V_y are neighbourhoods of x and y, respectively. They are clearly disjoint. Indeed, if there would exist $u \in V_x \cap V_y$ then

$$p_i(x - y) = p_i(x - u + u - y) \leq p_i(x - u) + p_i(u - y) < 2\varepsilon$$

which is a contradiction. □

Proposition 4.3.3. *A locally convex t.v.s. is Hausdorff if and only if its topology can be induced by a separating family of seminorms.*

Proof. Let (X, τ) be a locally convex t.v.s.. Then we know that there always exists a basis $\mathcal{N}$ of neighbourhoods of the origin in X consisting of open absorbing absolutely convex sets. Moreover, in Theorem 4.2.9, we have showed that $\tau = \tau_{\mathcal{P}}$ where $\mathcal{P}$ is the family of seminorms given by the Minkowski functionals of sets in $\mathcal{N}$, i.e. $\mathcal{P} := \{p_N : N \in \mathcal{N}\}$, and also that for each $N \in \mathcal{N}$ we have $N = \mathring{U}_{p_N}$.

Suppose that (X, τ) is also Hausdorff. Then Proposition 2.2.3 ensures that for any $x \in X$ with $x \neq o$ there exists a neighbourhood V of the origin in X s.t. $x \notin V$. This implies that there exists at least $N \in \mathcal{N}$ s.t. $x \notin N$ [2]. Hence, $x \notin N = \mathring{U}_{p_N}$ means that $p_N(x) \geq 1$ and so $p_N(x) \neq 0$, i.e. $\mathcal{P}$ is separating.

Conversely, if τ is induced by a separating family of seminorms $\mathcal{P}$, i.e. $\tau = \tau_{\mathcal{P}}$, then Lemma 4.3.2 ensures that X is Hausdorff.

□

Examples 4.3.4.

1. *Every normed space is a Hausdorff locally convex space, since every norm is a seminorm satisfying the separation property. Therefore, every Banach space is a complete Hausdorff locally convex space.*
2. *Every family of seminorms on a vector space containing a norm induces a Hausdorff locally convex topology.*
3. *Given an open subset Ω of $\mathbb{R}^d$ with the euclidean topology, the space $\mathcal{C}(\Omega)$ of real valued continuous functions on Ω with the so-called* topology of uniform convergence on compact sets *is a locally convex t.v.s.. This topology is defined by the family $\mathcal{P}$ of all the seminorms on $\mathcal{C}(\Omega)$ given by*
$$p_K(f) := \max_{x \in K} |f(x)|, \ \forall K \subset \Omega \ \text{compact}.$$
Moreover, $(\mathcal{C}(\Omega), \tau_{\mathcal{P}})$ is Hausdorff, because the family $\mathcal{P}$ is clearly separating. In fact, if $p_K(f) = 0$, $\forall K$ compact subsets of Ω then in particular $p_{\{x\}}(f) = |f(x)| = 0$ $\forall x \in \Omega$, which implies $f \equiv 0$ on Ω.
More generally, for any X locally compact we have that $\mathcal{C}(X)$ with the topology of uniform convergence on compact subsets of X is a locally convex Hausdorff t.v.s.

To introduce two other examples of l.c. Hausdorff t.v.s. we need to recall some standard general notations. Let $\mathbb{N}_0$ be the set of all non-negative integers. For any $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$ one defines $x^\alpha := x_1^{\alpha_1} \cdots x_d^{\alpha_d}$. For any $\beta \in \mathbb{N}_0^d$, the symbol D^β denotes the partial derivative of order $|\beta|$ where $|\beta| := \sum_{i=1}^d \beta_i$, i.e.

$$D^\beta := \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \cdots \partial x_d^{\beta_d}} = \frac{\partial^{\beta_1}}{\partial x_1^{\beta_1}} \cdots \frac{\partial^{\beta_d}}{\partial x_d^{\beta_d}}.$$

[2] Since $\mathcal{N}$ is a basis of neighbourhoods of the origin, $\exists\, M \in \mathcal{N}$ s.t. $M \subseteq V$. If x would belong to all elements of the basis then in particular it would be $x \in M$ and so also $x \in V$, contradiction.

Examples 4.3.5.

1. *Let $\Omega \subseteq \mathbb{R}^d$ open in the euclidean topology. For any $k \in \mathbb{N}_0$, let $\mathcal{C}^k(\Omega)$ be the set of all real valued $k-$*times continuously differentiable functions *on Ω, i.e. all the derivatives of f of order $\leq k$ exist (at every point of Ω) and are continuous functions in Ω. Clearly, when $k = 0$ we get the set $\mathcal{C}(\Omega)$ of all real valued continuous functions on Ω and when $k = \infty$ we get the so-called set of all* infinitely differentiable functions *or* smooth functions *on Ω. For any $k \in \mathbb{N}_0$, $\mathcal{C}^k(\Omega)$ (with pointwise addition and scalar multiplication) is a vector space over $\mathbb{R}$. The topology given by the following family of seminorms on $\mathcal{C}^k(\Omega)$:*

$$p_{m,K}(f) := \sup_{\substack{\beta \in \mathbb{N}_0^d \\ |\beta| \leq m}} \sup_{x \in K} \left|(D^\beta f)(x)\right|, \forall K \subseteq \Omega \text{ compact}, \forall m \in \{0, 1, \dots, k\},$$

 makes $\mathcal{C}^k(\Omega)$ into a l.c. Hausdorff t.v.s..
2. *The* Schwartz space *or* space of rapidly decreasing functions *on $\mathbb{R}^d$ is defined as the set $\mathcal{S}(\mathbb{R}^d)$ of all real-valued functions which are defined and infinitely differentiable on $\mathbb{R}^d$ and which have the additional property (regulating their growth at infinity) that all their derivatives tend to zero at infinity faster than any inverse power of x, i.e.*

$$\mathcal{S}(\mathbb{R}^d) := \left\{ f \in \mathcal{C}^\infty(\mathbb{R}^d) : \sup_{x \in \mathbb{R}^d} \left| x^\alpha D^\beta f(x) \right| < \infty, \ \forall \alpha, \beta \in \mathbb{N}_0^d \right\}.$$

 (For example, any smooth function f with compact support in $\mathbb{R}^d$ is in $\mathcal{S}(\mathbb{R}^d)$, since any derivative of f is continuous and supported on a compact subset of $\mathbb{R}^d$, so $x^\alpha(D^\beta f(x))$ has a maximum in $\mathbb{R}^d$ by the extreme value theorem.)

 The Schwartz space $\mathcal{S}(\mathbb{R}^d)$ is a vector space over $\mathbb{R}$ and the topology given by the family $\mathcal{Q}$ of seminorms on $\mathcal{S}(\mathbb{R}^d)$:

$$q_{\alpha,\beta}(f) := \sup_{x \in \mathbb{R}^d} \left| x^\alpha D^\beta f(x) \right|, \ \forall \alpha, \beta \in \mathbb{N}_0^d$$

 makes $\mathcal{S}(\mathbb{R}^d)$ into a l.c. Hausdorff t.v.s. (see Exercise Sheet 5).
 Note that $\mathcal{S}(\mathbb{R}^d)$ is a linear subspace of $\mathcal{C}^\infty(\mathbb{R}^d)$, but its topology $\tau_{\mathcal{Q}}$ on $\mathcal{S}(\mathbb{R}^d)$ is finer than the subspace topology induced on it by $\mathcal{C}^\infty(\mathbb{R}^d)$ (see Exercise Sheet 5).

4.4 The finest locally convex topology

In the previous sections we have seen how to generate topologies on a vector space which makes it into a locally convex t.v.s.. Among all of them, there is the finest one (i.e. the one having the largest number of open sets) which is usually called the *finest locally convex topology* on the given vector space.

Proposition 4.4.1. *The finest locally convex topology on a vector space X is the topology induced by the family of all seminorms on X and it is a Hausdorff topology.*