4.4 The finest locally convex topology

In the previous sections we have seen how to generate topologies on a vector space which makes it into a locally convex t.v.s.. Among all of them, there is the finest one (i.e. the one having the largest number of open sets) which is usually called the *finest locally convex topology* on the given vector space.

Proposition 4.4.1. *The finest locally convex topology on a non-trivial vector space X is the topology induced by the family of all seminorms on X and it is a Hausdorff topology.*

Proof.
Let us denote by $\mathcal{S}$ the family of all seminorms on the vector space X. By Theorem 4.2.9, we know that the topology $\tau_{\mathcal{S}}$ induced by $\mathcal{S}$ makes X into a locally convex t.v.s. We claim that $\tau_{\mathcal{S}}$ is the finest locally convex topology. In fact, if there was a finer locally convex topology τ (i.e. if $\tau_{\mathcal{S}} \subseteq \tau$ with (X, τ) locally convex t.v.s.) then Theorem 4.2.9 would give that τ is also induced by a family $\mathcal{P}$ of seminorms. But surely $\mathcal{P} \subseteq \mathcal{S}$ and so $\tau = \tau_{\mathcal{P}} \subseteq \tau_{\mathcal{S}}$ by definition of induced topology. Hence, $\tau = \tau_{\mathcal{S}}$.

It remains to show that $(X, \tau_{\mathcal{S}})$ is Hausdorff. By Lemma 4.3.2, it is enough to prove that $\mathcal{S}$ is separating. Let $x \in X \setminus \{o\}$ and let $\mathcal{B}$ be an algebraic basis of the vector space X containing x (its existence is guaranteed by Zorn's lemma). Define the linear functional $L : X \to \mathbb{K}$ as $L(x) = 1$ and $L(y) = 0$ for all $y \in \mathcal{B} \setminus \{x\}$. Then it is easy to see that $s := |L|$ is a seminorm, so $s \in \mathcal{S}$ and $s(x) \neq 0$, which proves that $\mathcal{S}$ is separating.[3] □

An alternative way of describing the finest locally convex topology on a vector space without using seminorms is the following:

Proposition 4.4.2. *The collection of all absorbing absolutely convex sets of a non-trivial vector space X is a basis of neighbourhoods of the origin for the finest locally convex topology on X.*

Proof. Let τ_{max} be the finest locally convex topology on X and $\mathcal{A}$ the collection of all absorbing absolutely convex sets of X. Since $\mathcal{A}$ fulfills all the properties required in Theorem 4.1.14, there exists a unique topology τ which makes X into a locally convex t.v.s.. Hence, by definition of finest locally convex

[3] Alternatively, we can show that $\mathcal{S}$ is separating by proving that there always exists a norm on X. In fact, let $\mathcal{B} = (b_i)_{i \in I}$ be an algebraic basis of X then for any $x \in X$ there exist a finite subset J of I and $\lambda_j \in \mathbb{K}$ for all $j \in J$ s.t. $x = \sum_{j \in J} \lambda_j b_j$ and so we can define $\|x\| := \max_{j \in J} |\lambda_j|$. Then it is easy to check that $\|\cdot\|$ is a norm on X. Hence, $\mathcal{S}$ always contains the norm $\|\cdot\|$ and so it is separating.

topology, $\tau \subseteq \tau_{max}$. On the other hand, (X, τ_{max}) is itself locally convex and so Theorem 4.1.14 ensures that has a basis $\mathcal{B}_{max}$ of neighbourhoods of the origin consisting of absorbing absolutely convex subsets of X. Then clearly $\mathcal{B}_{max}$ is contained in $\mathcal{A}$ and, hence, $\tau_{max} \subseteq \tau$. □

This result can be proved also using Proposition 4.4.1 and the correspondence between Minkowski functionals and absorbing absolutely convex subsets of X introduced in the Section 4.2 (see Exercise Sheet 5).

Proposition 4.4.3. *Every linear functional on a vector space X is continuous w.r.t. the finest locally convex topology on X.*

Proof. Let $L : X \to \mathbb{K}$ be a linear functional on a vector space X. For any $\varepsilon > 0$, we denote by $B_\varepsilon(0)$ the open ball in $\mathbb{K}$ of radius ε and center $0 \in \mathbb{K}$, i.e. $B_\varepsilon(0) := \{k \in \mathbb{K} : |k| < \varepsilon\}$. Then we have that $L^{-1}(B_\varepsilon(0)) = \{x \in X : |L(x)| < \varepsilon\}$. It is easy to verify that the latter is an absorbing absolutely convex subset of X and so, by Proposition 4.4.2, it is a neighbourhood of the origin in the finest locally convex topology on X. Hence L is continuous at the origin and so, by Proposition 2.1.15-3), L is continuous everywhere in X. □

4.5 Finite topology on a countable dimensional t.v.s.

In this section we are going to give an important example of finest locally convex topology on an infinite dimensional vector space, namely the *finite topology* on any countable dimensional vector space. For simplicity, we are going to focus on $\mathbb{R}-$vector spaces.

Definition 4.5.1. *Let X be an infinite dimensional vector space whose dimension is countable. The* finite topology *τ_f on X is defined as follows:*
$U \subseteq X$ is open in τ_f iff $U \cap W$ is open in the euclidean topology on W for all finite dimensional subspaces W of X.
Equivalently, if we fix an algebraic basis $\{x_n\}_{n\in\mathbb{N}}$ of X and if for any $n \in \mathbb{N}$ we set $X_n := span\{x_1, \ldots, x_n\}$ s.t. $X = \bigcup_{i=1}^{\infty} X_i$ and $X_1 \subseteq \ldots \subseteq X_n \subseteq \ldots$, then $U \subseteq X$ is open in τ_f iff $U \cap X_i$ is open in the euclidean topology on X_i for every $i \in \mathbb{N}$.

We actually already know a concrete example of countable dimensional space with the finite topology:

Example 4.5.2. *Let $n \in \mathbb{N}$ and $\underline{x} = (x_1, \dots, x_n)$. Denote by $\mathbb{R}[\underline{x}]$ the space of polynomials in the n variables $x_1, \dots, x_n$ with real coefficients and by*

$$\mathbb{R}_d[\underline{x}] := \{f \in \mathbb{R}[\underline{x}] | \deg f \leq d\}, d \in \mathbb{N}_0,$$

then $\mathbb{R}[\underline{x}] := \bigcup_{d=0}^{\infty} \mathbb{R}_d[\underline{x}]$. The finite topology τ_f on $\mathbb{R}[\underline{x}]$ is then given by: $U \subseteq \mathbb{R}[\underline{x}]$ is open in τ_f iff $\forall d \in \mathbb{N}_0$, $U \cap \mathbb{R}_d[\underline{x}]$ is open in $\mathbb{R}_d[\underline{x}]$ with the euclidean topology.

Theorem 4.5.3. *Let X be an infinite dimensional vector space whose dimension is countable endowed with the finite topology τ_f. Then:*
a) (X, τ_f) is a Hausdorff locally convex t.v.s.
b) τ_f is the finest locally convex topology on X

Proof.
a) We leave to the reader the proof of the fact that τ_f is compatible with the linear structure of X (see Exercise Sheet 6) and we focus instead on proving that τ_f is a locally convex topology. To this aim we are going to show that for any open neighbourhood U of the origin in (X, τ_f) there exists an open convex neighbourhood U' of the origin such that $U' \subseteq U$.

Let $\{x_i\}_{i \in \mathbb{N}}$ be an $\mathbb{R}$-basis for X and set $X_j := span\{x_1, \dots, x_j\}$ for any $j \in \mathbb{N}$. Fixed an open neighbourhood U of the origin in (X, τ_f), we are going to inductively construct an increasing sequence of convex subsets $(C_j)_{j \in \mathbb{N}}$ such that $C_j \subseteq U \cap X_j$ for any $j \in \mathbb{N}$. Indeed, we will show that

$$\forall\, j \in \mathbb{N}, \exists a_j \in \mathbb{R}_+ : C_j := \left\{\lambda_1 x_1 + \ldots + \lambda_j x_j \,|-a_i \leq \lambda_i \leq a_i\, ; i \in \{1, \dots, j\}\right\} \subseteq U \cap X_j. \tag{4.7}$$

Note that each C_j is a convex and closed in X_j as well as in X_{j+1}.

- $\underline{j = 1}$: Since $U \cap X_1$ is open in X_1, we have that there exists $a_1 \in \mathbb{R}_+$ such that $C_1 := \left\{\lambda_1 x_1 \mid -a_1 \leq \lambda_1 \leq a_1\right\} \subseteq U \cap X_1$, i.e. (4.7) holds.
- Inductive assumption: Fixed a natural number $n \geq 2$, suppose (4.7) holds for all $j \in \{1, \dots, n\}$, i.e. $\exists\, a_1, \dots, a_n \in \mathbb{R}_+$ s.t. $C_j \subseteq U \cap X_j$, $\forall j \in \{1, \dots, n\}$.
- $\underline{j = n + 1}$: We claim $\exists\; a_{n+1} \in \mathbb{R}_+$ such that $C_{n+1} \subseteq U \cap X_{n+1}$.
  If the claim does not hold, then $\forall\; a_{n+1} \in \mathbb{R}_+, \exists\; x \in C_{n+1}$ s.t. $x \notin U$. In particular, $\forall\; N \in \mathbb{N}\; \exists \lambda_1^N, \dots, \lambda_{n+1}^N \in \mathbb{R}$ such that $-a_i \leq \lambda_i^N \leq a_i$ for $i \in \{1, \dots, n\}$, $-\frac{1}{N} \leq \lambda_{n+1}^N \leq \frac{1}{N}$ and

$$x^N = \lambda_1^N x_1 + \ldots \lambda_n^N x_n + \lambda_{n+1}^N x_{n+1} \notin U.$$

  Hence, $\{x^N\}_{N \in \mathbb{N}}$ is bounded sequence of elements in $X_{n+1} \setminus U$. Therefore, we can find a convergent subsequence $\{x^{N_j}\}_{j \in \mathbb{N}}$ and we denote by

x its limit. Since $X_{n+1} \setminus U$ is closed in X_{n+1}, we have that $x \in X_{n+1} \setminus U$. However, x^N has the form $x^N = \underbrace{\lambda_1^N x_1 + \ldots + \lambda_n^N x_n}_{\in C_n} + \lambda_{n+1}^N x_{n+1}$, so its $(n+1)-$th component tends to 0 as $j \to \infty$ and, hence, $x \in C_n \subseteq U$ (since C_n is closed in X_{n+1}). This provides a contradiction, establishing the claim.

Now for any $n \in \mathbb{N}$ consider

$$D_n := \{\lambda_1 x_1 + \ldots + \lambda_n x_n \mid -a_i < \lambda_i < a_i \,; i \in \{1, \ldots, n\}\},$$

then $D_n \subset C_n \subseteq U \cap X_n$ is open and convex in X_n. Then $U' := \cup_{n \in \mathbb{N}} D_n$ is an open and convex neighbourhood of the origin in (X, τ_f) and $U' \subseteq U$.

b) Let us finally show that τ_f is actually the finest locally convex topology τ_{max} on X which gives in turn also that (X, τ_f) is Hausdorff. Since we have already showed that τ_f is a l.c. topology on X, clearly we have $\tau_f \subseteq \tau_{max}$ by definition of finest l.c. topology on X.

Conversely, let us consider $U \subseteq X$ open in τ_{max}. We want to show that U is open in τ_f, i.e. $W \cap U$ is open in the euclidean topology on W for any finite dimensional subspace W of X. Now each W inherits τ_{max} from X. Let us denote by τ_{max}^W the subspace topology induced by (X, τ_{max}) on W. By definition of subspace topology, we have that $W \cap U$ is open in τ_{max}^W. Moreover, by Proposition 4.4.1, we know that (X, τ_{max}) is a Hausdorff t.v.s. and so (W, τ_{max}^W) is a finite dimensional Hausdorff t.v.s. (by Proposition 2.1.15-1). Therefore, τ_{max}^W has to coincide with the euclidean topology by Theorem 3.1.1 and, consequently, $W \cap U$ is open w.r.t. the euclidean topology on W. □

4.6 Continuity of linear mappings on locally convex spaces

In the context of l.c. spaces, it is natural to ask whether the continuity of linear maps can be characterized via seminorms. In this section, we in fact present a necessary and sufficient condition for the continuity of a linear map between two l.c. spaces only bearing on the seminorms inducing the two topologies.

For simplicity, let us start with linear functionals on a l.c. space. Recall that for us $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ endowed with the euclidean topology given by the absolute value $|\cdot|$. In this section, for any $\varepsilon > 0$ we denote by $B_\varepsilon(0)$ the open ball in $\mathbb{K}$ of radius ε and center $0 \in \mathbb{K}$ i.e. $B_\varepsilon(0) := \{k \in \mathbb{K} : |k| < \varepsilon\}$.

Proposition 4.6.1. *Let τ be a locally convex topology on a vector space X generated by a directed family $\mathcal{Q}$ of seminorms on X and $L : X \to \mathbb{K}$ linear. Then L is a τ-continuous iff there exists $q \in \mathcal{Q}$ s.t. L is q-continuous, i.e.*

$$\exists q \in \mathcal{Q}, \exists C > 0 \;\; s.t. \, |L(x)| \leq C q(x), \, \forall \, x \in X. \tag{4.8}$$

Proof.
Let us first observe that since X and $\mathbb{K}$ are both t.v.s. by Proposition 2.1.15-3) the continuity of L is equivalent to its continuity at the origin. Therefore, it is enough to prove the criterion for the continuity of L at the origin.

The τ-continuity of L at the origin in X means that for any $\varepsilon > 0$ $L^{-1}(B_\varepsilon(0)) = \{x \in X : |L(x)| < \varepsilon\}$ is an open neighbourhood of the origin in (X, τ). Since the family $\mathcal{Q}$ inducing τ is directed, a basis of neighbourhood of the origin in (X, τ) is given by $\mathcal{B}_d$ as in (4.5). Therefore, L is τ-continuous at the origin in X if and only if $\forall\, \varepsilon > 0$, $\exists B \in \mathcal{B}_d$ s.t. $B \subseteq L^{-1}(B_\varepsilon(0))$, i.e.

$$\forall\, \varepsilon > 0,\ \exists\ q \in \mathcal{Q},\ \exists\ r > 0 \ \text{ s.t. } \ r\mathring{U}_q \subseteq L^{-1}(B_\varepsilon(0)). \tag{4.9}$$

[4] ($\Rightarrow$) Suppose L is τ-continuous at the origin in X then (4.9) implies that L is $q-$continuous at the origin, because $r\mathring{U}_q$ is clearly an open neighbourhood of the origin in X w.r.t. the topology generated by the single seminorm q.

($\Leftarrow$) Suppose that there exists $q \in \mathcal{Q}$ s.t. L is q-continuous in X. Then, since τ is the topology induced by the whole family $\mathcal{Q}$ which is finer than the one induced by the single seminorm q, we clearly have that L is also $\tau-$continuous. □

By using this result together with Proposition 4.2.14 we get the following.

Corollary 4.6.2. *Let τ be a locally convex topology on a vector space X generated by a family $\mathcal{P} := \{p_i\}_{i \in I}$ of seminorms on X. Then $L : X \to \mathbb{K}$ is a τ-continuous linear functional iff there exist $n \in \mathbb{N}$, $i_1, \dots, i_n \in I$ such that L is $\big(\max_{k=1,\dots,n} p_{i_k}\big)$-continuous, i.e.*

$$\exists\, n \in \mathbb{N}, \exists\, i_1, \dots, i_n \in I, \exists\, C > 0 \ \textit{ s.t. } \ |L(x)| \leq C \max_{k=1,\dots,n} p_{i_k}(x),\ \forall\, x \in X.$$

The proof of Proposition 4.6.1 can be easily modified to get the following more general criterion for the continuity of any linear map between two locally convex spaces.

Theorem 4.6.3. *Let X and Y be two locally convex t.v.s. whose topologies are respectively generated by the families $\mathcal{P}$ and $\mathcal{Q}$ of seminorms on X. Then $f : X \to Y$ linear is continuous iff*

$$\forall\, q \in \mathcal{Q},\ \exists\, n \in \mathbb{N}, \exists\, p_1, \dots, p_n \in \mathcal{P}, \exists\, C > 0 \,:\, q(f(x)) \leq C \max_{i=1,\dots,n} p_i(x),\ \forall\, x \in X.$$

Proof. (Exercise Sheet 6) □

[4] Alternative proof: By simply observing that $|L|$ is a seminorm and by using Proposition 4.2.10, one gets that (4.8) is equivalent to (4.9) and so to the q-continuity of L.