Theorem 5.1.3 (Analytic form of HBT for sublinear functionals)**.** *Let p be a sublinear functional on a vector space X over $\mathbb{R}$, M a linear subspace of X, and f a linear functional on M such that*

$$f(x) \leq p(x), \forall\, x \in M.$$

There exists a linear functional $\tilde{f}$ on X such that $\tilde{f}(x) = f(x), \forall\, x \in M$ and

$$\tilde{f}(x) \leq p(x), \forall\, x \in X.$$

We are not going to show this version as it is usually proved in any Functional Analysis text book but we show instead how Theorem 5.1.2 can be deduced from it.

Proof. Theorem 5.1.3 $\Rightarrow$ Theorem 5.1.2

1) Case $\mathbb{K} = \mathbb{R}$

Let X be a t.v.s. over $\mathbb{R}$, N a linear subspace of X, and Ω a non-empty open convex subset of X such that $N \cap \Omega = \emptyset$. Fixed $n_0 \in N$ and $w_0 \in \Omega$, let $x_0 := n_0 - w_0$. Note that $x_0 \neq o$ otherwise $n_0 = w_0 \in N \cap \Omega$, which contradicts the assumption that $N \cap \Omega = \emptyset$. Then $C := \Omega - N + x_0$ is an open convex neighbourhood of the origin o in X. In fact, $C = \bigcup_{n \in N}(\Omega - n + x_0)$ is clearly open and convex as union of such sets (recall that the topology of a t.v.s. is translation invariant) and $o = w_0 - n_0 + x_0 \in C$. Then the Minkowski functional p_C associated to C is a sublinear functional on X which assumes finite non-negative values. Indeed:

- since C is absorbing, for all $x \in X$ we have that the set $\{h > 0 : x \in hC\}$ is non-empty and so $0 \leq p_C(x) < \infty$
- the convexity of C ensures the subadditivity of p_C
- for all $\mu > 0$ and all $x \in X$, we have

$$\begin{aligned} p_C(\mu x) &= \inf\{\lambda > 0 : \mu x \in \lambda C\} = \inf\{\lambda > 0 : x \in \frac{\lambda}{\mu} C\} \\ &= \inf\{\mu \frac{\lambda}{\mu} > 0 : x \in \frac{\lambda}{\mu} C\} = \mu \inf\{h > 0 : x \in hC\} = \mu p_C(x). \end{aligned}$$

Moreover, $x_0 \notin C$ (otherwise there would exist $w \in \Omega, n \in N$ such that $x_0 = w - n + x_0$, i.e. $w = n \in N \cap \Omega$ which would contradict the assumption that $N \cap \Omega = \emptyset$). This implies that $p_C(x_0) \geq 1$, because otherwise there exists $0 \leq \lambda \leq 1$ such that $x_0 \in \lambda C \subseteq C$[1]which yields a contraction. Since C is open

[1]For all $c \in C$ we have $c = w - n + x_0$ for some $w \in \Omega$, $n \in N$, and so for all $0 \leq \lambda \leq 1$

$$\begin{aligned} \lambda c &= \lambda w - \lambda n + \lambda x_0 = \lambda w - \lambda n + \lambda n_0 - \lambda w_0 \\ &= \lambda w + (1-\lambda) w_0 - w_0 + n_0 - n_0 - \lambda n + \lambda n_0 \\ &= \underbrace{\lambda w + (1-\lambda) w_0}_{\in \Omega} + x_0 - \underbrace{(n_0 + \lambda n - \lambda n_0)}_{\in N} \in C. \end{aligned}$$

and the scalar multiplication is continuous, we also have that for any $x \in C$ there exists $0 < \mu < 1$ such that $x \in \mu C$ and so $p_C(x) \leq \mu < 1$.

Now let M be the real vector space spanned by x_0 and consider $f : M \to \mathbb{R}$ defined by $f(tx_0) := t$ for all $t \in \mathbb{R}$. Then $f(m) \leq p_C(m)$ for all $m \in M$, because if $t > 0$ then $f(tx_0) = t \leq tp_C(x_0) = p_C(tx_0)$ and if $t \leq 0$ then $f(tx_0) = t \leq 0 \leq p_C(tx_0)$. Therefore, we can apply Theorem 5.1.3 which ensures the existence of a $\mathbb{R}-$linear functional $\tilde{f} : X \to \mathbb{R}$ such that $\tilde{f} \restriction_M = f$ and $\tilde{f}(x) \leq p_C(x)$ for all $x \in X$.

Hence, for any $n \in N$ and $w \in \Omega$ we have that

$$\tilde{f}(w - n + x_0) \leq p_C(\underbrace{w - n + x_0}_{\in C}) < 1$$

and so $\tilde{f}(w) - \tilde{f}(n) < 1 - \tilde{f}(x_0) \overset{x_0 \in M}{=} 1 - f(x_0) = 1 - 1 = 0$, i.e.

$$\tilde{f}(w) < \tilde{f}(n), \forall n \in N, \ w \in \Omega \tag{5.5}$$

This implies that $\tilde{f}(N) = \{0\}$. In fact, if there existed $0 \neq r \in \tilde{f}(N)$ then $\tilde{f}(N) = \mathbb{R}$ and so $\tilde{f}(N) \cap \tilde{f}(\Omega) \neq \emptyset$, that is, $\exists y \in \tilde{f}(N) \cap \tilde{f}(\Omega)$ i.e. $\exists n \in N, w \in \Omega$ s.t. $y = \tilde{f}(w) = \tilde{f}(n)$ contradicting (5.5).

Taking $H := \{x \in X : \tilde{f}(x) = 0\}$ yields the conclusion. In fact, H is a real hyperplane in X such that:

- $N \subset H$, since we showed $\tilde{f}(N) = \{0\}$
- $H \cap \Omega = \emptyset$, because if there was $x \in H \cap \Omega$ then $0 = \tilde{f}(x) \overset{ineq}{<} \tilde{f}(o) = 0$ which is a contraction.

2) Case $\mathbb{K} = \mathbb{C}$

Let X be a t.v.s. over $\mathbb{C}$ and N a complex linear subspace of X. Looking at X and N as linear spaces over $\mathbb{R}$, we can use the proof above to get that there exists a $\mathbb{R}-$linear functional $\tilde{f} : X \to \mathbb{R}$ such that $\tilde{f}(x) \leq p_C(x)$ for all $x \in X$, (5.5) holds and so $\tilde{f}(N) = \{o\}$.

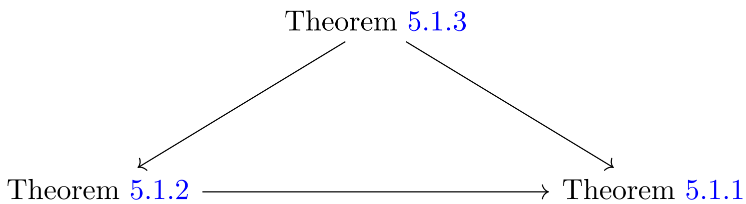
Define $g : X \to \mathbb{C}$ by $g(x) := \tilde{f}(x) - i\tilde{f}(ix), \forall\, x \in X$. Then g is a $\mathbb{C}$-linear functional on X and $H := \{x \in X : g(x) = 0\}$ a complex hyperplane s.t.

- $N \subset H$, because for any $n \in N$ we have that $in \in N$ and so $g(n) = \tilde{f}(n) - i\tilde{f}(in) = 0$ as $\tilde{f}(N) = \{0\}$.
- $H \cap \Omega = \emptyset$, because if there was $x \in H \cap \Omega$ then $0 = g(x) = \tilde{f}(x) - i\tilde{f}(ix)$ and so $\underbrace{\tilde{f}(x)}_{\in \mathbb{R}} = i\underbrace{\tilde{f}(ix)}_{\in \mathbb{R}}$ which implies that $\tilde{f}(ix) = 0$.

  Hence, $\tilde{f}(x) = 0 = \tilde{f}(o) \overset{(5.5)}{>} \tilde{f}(x)$ which is a contradiction.

□

Summing up we have that



5.2 Applications of Hahn-Banach theorem

The Hahn-Banach theorem is frequently applied in analysis, algebra and geometry, as will be seen in the forthcoming course. We will briefly indicate in this section some applications of this theorem to problems of separation of convex sets and to the multivariate moment problem. From now on we will focus on t.v.s. over the field of real numbers.

5.2.1 Separation of convex subsets of a real t.v.s.

Let X t.v.s. over the field of real numbers and H be a closed affine hyperplane of X. We say that two disjoint subsets A and B of X are *separated* by H if A is contained in one of the two closed half-spaces determined by H and B is contained in the other one. We can express this property in terms of functionals. Indeed, since $H = L^{-1}(\{a\})$ for some $L : X \to \mathbb{R}$ linear not identically zero and some $a \in \mathbb{R}$, we can write that A and B are separated by H if and only if:

$$\exists\, a \in \mathbb{R} \text{ s.t. } L(A) \geq a \text{ and } L(B) \leq a.$$

where for any $S \subseteq X$ the notation $L(S) \leq a$ simply means $\forall s \in S, L(s) \leq a$ (and analogously for $\geq, <, >, =, \neq$).
We say that A and B are *strictly separated* by H if at least one of the two inequalities is strict. (Note that there are several definition in literature for the strict separation but for us it will be just the one defined above) In the present subsection we would like to investigate whether one can separate, or strictly separate, two disjoint convex subsets of a real t.v.s..

Proposition 5.2.1. *Let X be a t.v.s. over the real numbers and A, B two disjoint nonempty convex subsets of X.*
a) If A is open, then there exists a closed affine hyperplane H of X separating A and B, i.e. there exists $a \in \mathbb{R}$ and a functional $L : X \to \mathbb{R}$ linear not identically zero s.t. $L(A) \geq a$ and $L(B) \leq a$.

b) *If A and B are both open, the hyperplane H can be chosen so as to strictly separate A and B, i.e. there exists $a \in \mathbb{R}$ and $L : X \to \mathbb{R}$ linear not identically zero s.t. $L(A) \geq a$ and $L(B) < a$.*
c) *If A is a cone and B is open, then a can be chosen to be zero, i.e. there exists $L : X \to \mathbb{R}$ linear not identically zero s.t. $L(A) \geq 0$ and $L(B) < 0$.*

Proof.

a) Consider the set $A - B := \{a - b : a \in A, b \in B\}$. Then: $A - B$ is an open subset of X as it is the union of the open sets $A - y$ as y varies over B; $A - B$ is convex as it is the Minkowski sum of the convex sets A and $-B$; and $o \notin (A-B)$ because if this was the case then there would be at least a point in the intersection of A and B which contradicts the assumption that they are disjoint. By applying Theorem 5.1.2 to $N = \{o\}$ and $U = A - B$ we have that there is a closed hyperplane H of X which does not intersect $A-B$ (and passes through the origin) or, which is equivalent, there exists a linear form f on X not identically zero such that $f(A-B) \neq 0$. Then there exists a linear form L on X not identically zero such that $L(A - B) > 0$ (in the case $f(A - B) < 0$ just take $L := -f$), i.e.

$$\forall\, x \in A,\, \forall\, y \in B,\;\; L(x) > L(y). \tag{5.6}$$

Since $B \neq \emptyset$ we have that $a := \inf_{x \in A} L(x) > -\infty$. Then (5.6) implies that $L(B) \leq a$ and we clearly have $L(A) \geq a$.

b) Let now both A and B be open convex and nonempty disjoint subsets of X. By part a) we have that there exists $a \in \mathbb{R}$ and $L : X \to \mathbb{R}$ linear not identically zero s.t. $L(A) \geq a$ and $L(B) \leq a$. Suppose that there exists $b \in B$ s.t. $L(b) = a$. Since B is open, for any $x \in X$ there exists $\varepsilon > 0$ s.t. for all $t \in [0, \varepsilon]$ we have $b + tx \in B$. Therefore, as $L(B) \leq a$, we have that

$$L(b + tx) \leq a, \forall\, t \in [0, \varepsilon]. \tag{5.7}$$

Now fix $x \in X$, consider the function $f(t) := L(b + tx)$ for all $t \in \mathbb{R}$ whose first derivative is clearly given by $f'(t) = L(x)$ for all $t \in \mathbb{R}$. Then (5.7) means that $t = 0$ is a point of local maximum for f and so $f'(0) = 0$ i.e. $L(x) = 0$. As x is an arbitrary point of x, we get $L \equiv 0$ on X which is a contradiction. Hence, $L(B) < a$.

c) Let now A be a nonempty convex cone of X and B an open convex nonempty subset of X s.t. $A \cap B = \emptyset$. By part a) we have that there exists $a \in \mathbb{R}$ and $L : X \to \mathbb{R}$ linear not identically zero s.t. $L(A) \geq a$ and $L(B) \leq a$. Since A is a cone, for any $t > 0$ we have that $tA \subseteq A$ and so $tL(A) = L(tA) \geq a$ i.e. $L(A) \geq \frac{a}{t}$. This implies that $L(A) \geq \inf_{t>0} \frac{a}{t} = 0$.

Moreover, part a) also gives that $L(B) < L(A)$. Therefore, for any $t > 0$ and any $x \in A$, we have in particular $L(B) < L(tx) = tL(x)$ and so $L(B) \leq \inf_{t>0} tL(x) = 0$. Since B is also open, we can exactly proceed as in part b) to get $L(B) < 0$.

□

Let us show now two interesting consequences of this result which we will use in the following subsection.

Corollary 5.2.2. *Let (X, τ) be a locally convex t.v.s. over $\mathbb{R}$ endowed. If C is a nonempty closed convex cone in X and $x_0 \in X \setminus C$ then there exists a linear functional $L : X \to \mathbb{R}$ non identically zero s.t. $L(C) \geq 0$ and $L(x_0) < 0$.*

Proof. As C is closed in (X, τ) and $x_0 \in X \setminus C$, we have that $X \setminus C$ is an open neighbourhood of x_0. Then the local convexity of (X, τ) guarantees that there exists an open convex neighbourhood V of x_0 s.t. $V \subseteq X \setminus C$ i.e. $V \cap C = \emptyset$. By Proposition 5.2.1-c), we have that there exists $L : X \to \mathbb{R}$ linear not identically zero s.t. $L(C) \geq 0$ and $L(V) < 0$, in particular $L(x_0) < 0$. □

Before giving the second corollary, let us introduce some notations. Given a convex cone C in a t.v.s. (X, τ) we define the first and the second dual of C w.r.t. τ respectively as follows:

$$C_\tau^\vee := \{\ell : X \to \mathbb{R} \text{ linear} \,|\, \ell \text{ is} \, \tau-\text{continuous and } \ell(C) \geq 0\}$$

$$C_\tau^{\vee\vee} := \{x \in X \,|\, \forall \, \ell \in C_\tau^\vee, \, \ell(x) \geq 0\}.$$

Corollary 5.2.3. *Let X be real vector space endowed with the finest locally convex topology φ. If C is a nonempty convex cone in X, then $\overline{C}^\varphi = C_\varphi^{\vee\vee}$.*

Proof. Let us first observe that $\overline{C}^\varphi \subseteq C_\varphi^{\vee\vee}$. Indeed, if $x \in \overline{C}^\varphi$ then for any $\ell \in C_\varphi^\vee$ we have by definition of first dual of C that $\ell(x) \geq 0$. Hence, $x \in C_\varphi^{\vee\vee}$.

Conversely, suppose there exists $x_0 \in C_\varphi^{\vee\vee} \setminus \overline{C}^\varphi$. By Corollary 5.2.2, there exists a linear functional $L : X \to \mathbb{R}$ non identically zero s.t. $L(\overline{C}^\varphi) \geq 0$ and $L(x_0) < 0$. As $L(C) \geq 0$ and every linear functional is $\varphi-$continuous, we have $L \in C_\varphi^\vee$. This together with the fact that $L(x_0) < 0$ give $x_0 \notin C_\varphi^{\vee\vee}$, which is a contradiction. Hence, $\overline{C}^\varphi = C_\varphi^{\vee\vee}$. □