5.2.2 Multivariate real moment problem

The moment problem has been first introduced by Stieltjes in 1894 (see [12]) for the case $K = [0, +\infty)$, as a mean of studying the analytic behaviour of continued fractions. Since then it has been largely investigated in a wide range of subjects, but the theory is still far from being up to the demand of applications. In this section we are going to give a very brief introduction to this problem in the finite dimensional setting but for more detailed surveys on this topics see e.g. [1, 8, 9].

Let μ be a nonnegative Borel measure defined on $\mathbb{R}$. The *$n-$th moment of μ* is defined as

$$m_n^\mu := \int_{\mathbb{R}} x^n \mu(dx)$$

If all moments of μ exist and are finite, then $(m_n^\mu)_{n=0}^\infty$ is called the *moment sequence* of μ. The moment problem addresses exactly the inverse question.

Definition 5.2.4 (Univariate real $K-$moment problem)**.**
Given a a closed subset K of $\mathbb{R}$ and a sequence $m := (m_n)_{n=0}^\infty$ with $m_n \in \mathbb{R}$, does there exists a nonnegative finite Radon measure μ having m as its moment sequence and support supp(μ) contained in K, i.e. such that

$$m_n = \int_K x^n \mu(dx), \quad \forall n \in \mathbb{N}_0 \quad \textit{and} \quad \textit{supp}(\mu) \subseteq K?$$

*If such a measure exists, we say that μ is a K-*representing measure *for m and that it is a solution to the $K-$moment problem for m.*

Recall that a *Radon measure* μ on a Hausdorff topological space X is a non-negative Borel measure which is locally finite (i.e. every point of X has a neighbourhood of finite measure) and inner regular (i.e. for each Borel measurable set B in X we have $\mu(B) = \sup\{\mu(K) : K \subseteq B, \text{ compact}\}$).

To any sequence $m := (m_n)_{n=0}^\infty$ of real numbers we can always associate the so-called *Riesz' functional* defined by:

$$\begin{array}{lll} L_m\colon & \mathbb{R}[x] & \to \mathbb{R} \\ & p(x) := \sum\limits_{n=0}^{N} p_n x^n & \mapsto L_m(p) := \sum\limits_{n=0}^{N} p_n m_n. \end{array}$$

If μ is a $K-$representing measure for m, then

$$L_m(p) = \sum_{n=0}^{N} p_n m_n = \sum_{n=0}^{N} p_n \int_K x^n \mu(dx) = \int_K p(x)\mu(dx).$$

Hence, there is the following bijective correspondence between the set $\mathbb{R}^{\mathbb{N}_0}$ of all sequences of real numbers and the set $(\mathbb{R}[x])^*$ of all linear functional from $\mathbb{R}[x]$ to $\mathbb{R}$

$$\begin{array}{rcl} \mathbb{R}^{\mathbb{N}_0} & \to & (\mathbb{R}[x])^* \\ (m_n)_{n\in\mathbb{N}_0} & \mapsto & L_m \\ & & \\ (L(x^n))_{n\in\mathbb{N}_0} & \leftarrow\!\shortmid & L \end{array}$$

which allows us to reformulate the univariate $K-$moment problem in terms of linear functionals.

Definition 5.2.5 (Univariate real $K-$moment problem)**.**
Given a closed subset K of $\mathbb{R}^d$ and a linear functional $L : \mathbb{R}[x] \to \mathbb{R}$, does there exists a nonnegative finite Radon measure μ s.t.

$$L(p) = \int_{\mathbb{R}^d} p(x)\mu(dx),\ \forall p \in \mathbb{R}[x] \quad \text{and} \quad supp(\mu) \subseteq K?$$

This formulation clearly shows us how to pose the problem in higher dimensions, but before that let us fix some notations. Let $d \in \mathbb{N}$ and let $\mathbb{R}[\underline{x}]$ be the ring of polynomials with real coefficients and d variables $\underline{x} := (x_1, \dots, x_d)$.

Definition 5.2.6 (Multivariate real $K-$moment problem)**.**
Given a closed subset K of $\mathbb{R}^d$ and a linear functional $L : \mathbb{R}[\underline{x}] \to \mathbb{R}$, does there exists a nonnegative finite Borel measure μ s.t.

$$L(p) = \int_{\mathbb{R}^d} p(\underline{x})\mu(d\underline{x}),\ \forall p \in \mathbb{R}[\underline{x}]$$

and $supp(\mu) \subseteq K$?

If such a measure exists, we say that μ is a K-representing measure *for L and that it is a solution to the $K-$moment problem for L.*

A necessary condition for the existence of a solution to the $K-$moment problem for the linear functional L is clearly that L is nonnegative on

$$Psd(K) := \{p \in \mathbb{R}[\underline{x}] : p(\underline{x}) \geq 0, \forall x \in K\}.$$

In fact, if there exists a $K-$representing measure μ for L then for all $p \in Psd(K)$ we have

$$L(p) = \int_{\mathbb{R}^d} p(\underline{x})\mu(d\underline{x}) = \int_K p(\underline{x})\mu(d\underline{x}) \geq 0$$

since μ is nonnegative and supported on K and p is nonnegative on K.

It is then natural to ask if the nonnegative of L on $Psd(K)$ is also sufficient. The answer is positive and it was established by Riesz in 1923 for $d = 1$ and by Haviland for any $d \geq 2$.

Theorem 5.2.7 (Riesz-Haviland Theorem)**.** *Let K be a closed subset of $\mathbb{R}^d$ and $L : \mathbb{R}[\underline{x}] \to \mathbb{R}$ be linear. L has a $K-$representing measure if and only if $L(Psd(K)) \geq 0$.*

Note that this theorem provides a complete solution for the $K-$ moment problem but it is quite unpractical! In fact, it reduces the $K-$moment problem to the problem of classifying all polynomials which are nonnegative on a prescribed closed subset K of $\mathbb{R}^d$ i.e. to characterize $Psd(K)$. This is actually a hard problem to be solved for general K and it is a core question in real algebraic geometry. For example, if we think of the case $K = \mathbb{R}^d$ then for $d = 1$ we know that $Psd(K) = \sum \mathbb{R}[\underline{x}]^2$, where $\sum \mathbb{R}[\underline{x}]^2$ denotes the set of squares of polynomials. However, for $d \geq 2$ this equality does not hold anymore as it was proved by Hilbert in 1888. It is now clear that to make the conditions of the Riesz-Haviland theorem actually checkable we need to be able to write/approximate a non-negative polynomial on K by polynomials whose non-negativity is "more evident", i.e. sums of squares or elements of quadratic modules of $\mathbb{R}[\underline{x}]$. For a special class of closed subsets of $\mathbb{R}^d$ we actually have such representations and we can get better conditions than the ones of Riesz-Haviland type to solve the $K-$moment problem.

Definition 5.2.8. *Given a finite set of polynomials $S := \{g_1, \ldots, g_s\}$, we call the* basic closed semialgebraic set *generated by S the following*

$$K_S := \{\underline{x} \in \mathbb{R}^d : g_i(\underline{x}) \geq 0,\ i = 1, \ldots, s\}.$$

Definition 5.2.9. *A subset M of $\mathbb{R}[\underline{x}]$ is said to be a* quadratic module *if $1 \in M$, $M + M \subseteq M$ and $h^2 M \subseteq M$ for any $h \in \mathbb{R}[\underline{x}]$.*

Note that each quadratic module is a convex cone in $\mathbb{R}[\underline{x}]$.

Definition 5.2.10. *A quadratic module M of $\mathbb{R}[\underline{x}]$ is called* Archimedean *if there exists $N \in \mathbb{N}$ s.t. $N - (\sum_{i=1}^d x_i^2) \in M$.*

For $S := \{g_1, \ldots, g_s\}$ finite subset of $\mathbb{R}[\underline{x}]$, we define the *quadratic module generated by S* to be:

$$M_S := \left\{ \sum_{i=0}^{s} \sigma_i g_i : \sigma_i \in \sum \mathbb{R}[\underline{x}]^2, i = 0, 1, \ldots, s \right\},$$

where $g_0 := 1$.

Remark 5.2.11. *Note that $M_S \subseteq Psd(K_S)$ and M_S is the smallest quadratic module of $\mathbb{R}[\underline{x}]$ containing S.*

Consider now the finite topology on $\mathbb{R}[\underline{x}]$ (see Definition 4.5.1) which we have proved to be the finest locally convex topology on this space (see Proposition 4.5.3) and which we therefore denote by φ. By Corollary 5.2.3, we get that

$$\overline{M_S}^{\varphi} = (M_S)^{\vee\vee}_{\varphi} \tag{5.8}$$

Moreover, the *Putinar Positivstellesatz* (1993), a milestone result in real algebraic geometry, provides that if M_S is Archimedean then

$$Psd(K_S) \subseteq \overline{M_S}^{\varphi}. \tag{5.9}$$

Note that M_S is Archimedean implies that K_S is compact while the converse is in general not true (see e.g. [9]).

Combining (5.8) and (5.9), we get the following result.

Proposition 5.2.12. *Let $S := \{g_1, \dots, g_s\}$ be a finite subset of $\mathbb{R}[\underline{x}]$ and $L : \mathbb{R}[\underline{x}] \to \mathbb{R}$ linear. Assume that M_S is Archimedean. Then there exists a K_S-representing measure μ for L if and only if $L(M_S) \geq 0$, i.e. $L(h^2 g_i) \geq 0$ for all $h \in \mathbb{R}[\underline{x}]$ and for all $i \in \{1, \dots, s\}$.*

Proof. Suppose that $L(M_S) \geq 0$ and let us consider the finite topology φ on $\mathbb{R}[\underline{x}]$. Then the linear functional L is φ-continuous and so $L \in (M_S)^{\vee}_{\varphi}$. Moreover, as M_S is assumed to be Archimedean, we have

$$Psd(K_S) \overset{(5.9)}{\subseteq} \overline{M_S}^{\varphi} \overset{(5.8)}{=} (M_S)^{\vee\vee}_{\varphi}.$$

Since any $p \in Psd(K_S)$ is also an element of $(M_S)^{\vee\vee}_{\varphi}$, we have that for any $\ell \in (M_S)^{\vee}_{\varphi}$, $\ell(Psd(K_S)) \geq 0$ and in particular $L(Psd(K_S)) \geq 0$. Hence, by Riesz-Haviland theorem we get the existence of a K_S-representing measure μ for L.

Conversely, suppose that the there exists a K_S-representing measure μ for L. Then for all $p \in M_S$ we have in particular that

$$L(p) = \int_{\mathbb{R}^d} p(\underline{x}) \mu(d\underline{x})$$

which is nonnegative as μ is a nonnegative measure supported on K_S and $p \in M_S \subseteq Psd(K_S)$. □

From this result and its proof we understand that whenever we know that $Psd(K_S) \subseteq \overline{M_S}^{\varphi}$, we need to check only that $L(M_S) \geq 0$ to find out whether or not there exists a solution for the K_S-moment problem for L. Then it makes sense to look for closure results of this kind in the case when M_S is not Archimedean and so we cannot apply the Putinar Positivstellesatz. Actually, whenever we can find a locally convex topology τ on $\mathbb{R}[\underline{x}]$ for which $Psd(K_S) \subseteq \overline{M_S}^{\tau}$, the conditions $L(M_S) \geq 0$ is necessary and sufficient for the existence of a solution of the K_S-moment problem for any $\tau-$continuous linear functional L on $\mathbb{R}[\underline{x}]$ (see [2]). This relationship between the closure of quadratic modules and the representability of functionals continuous w.r.t. locally convex topologies started a new research line in the study of the moment problem which is still bringing interesting results.