**Example 1.1.13.** *The open sets of a topological space other than the empty set always form a basis of neighbourhoods.*

**Theorem 1.1.14.** *Given a topological space $X$ and a point $x \in X$, a basis of open neighbourhoods $\mathcal{B}(x)$ satisfies the following properties.*
**(B1)** *For any $U \in \mathcal{B}(x)$, $x \in U$.*
**(B2)** *For any $U_1, U_2 \in \mathcal{B}(x)$, $\exists U_3 \in \mathcal{B}(x)$ s.t. $U_3 \subseteq U_1 \cap U_2$.*
**(B3)** *If $y \in U \in \mathcal{B}(x)$, then $\exists W \in \mathcal{B}(y)$ s.t. $W \subseteq U$.*
*Viceversa, if for each point $x$ in a set $X$ we are given a collection of subsets $\mathcal{B}_x$ fulfilling the properties (B1), (B2) and (B3) then there exists a unique topology $\tau$ s.t. for each $x \in X$, $\mathcal{B}_x$ is a basis of neighbourhoods of $x$, i.e. $\mathcal{B}_x \equiv \mathcal{B}(x)$, $\forall x \in X$.*

*Proof.* The proof easily follows by using Theorem 1.1.10. □

The previous theorem gives a further way of introducing a topology on a set. Indeed, starting from a basis of neighbourhoods of $X$, we can define a topology on $X$ by setting that a set is open iff whenever it contains a point it also contains a basic neighbourhood of the point. Thus a topology on a set $X$ is uniquely determined by a basis of neighbourhoods of each of its points.

### 1.1.2 Comparison of topologies

Any set $X$ may carry several different topologies. When we deal with topological vector spaces, we very often encounter this situation, i.e. a a vector space carrying several topologies all compatible with the linear structure in a sense that is going to be specified soon (in Chapter 2). In this case, it is convenient being able to compare topologies.

**Definition 1.1.15.** *Let $\tau$, $\tau'$ be two topologies on the same set $X$. We say that $\tau$ is* coarser *(or* weaker*) than $\tau'$, in symbols $\tau \subseteq \tau'$, if every subset of $X$ which is open for $\tau$ is also open for $\tau'$, or equivalently, if every neighborhood of a point in $X$ w.r.t. $\tau$ is also a neighborhood of that same point in the topology $\tau'$. In this case $\tau'$ is said to be* finer *(or* stronger*) than $\tau'$.*

Denote by $\mathcal{F}(x)$ and $\mathcal{F}'(x)$ the filter of neighbourhoods of a point $x \in X$ w.r.t. $\tau$ and w.r.t. $\tau'$, respectively. Then: $\tau$ is coarser than $\tau'$ iff for any point $x \in X$ we have $\mathcal{F}(x) \subseteq \mathcal{F}'(x)$ (this means that every subset of $X$ which belongs to $\mathcal{F}(x)$ also belongs to $\mathcal{F}'(x)$).

Two topologies $\tau$ and $\tau'$ on the same set $X$ *coincide* when they give the same open sets or the same closed sets or the same neighbourhoods of each point; equivalently, when $\tau$ is both coarser and finer than $\tau'$. Two basis of neighbourhoods in $X$ set are *equivalent* when they define the same topology.

**Remark 1.1.16.** *Given two topologies on the same set, it may very well happen that none is finer than the other. If it is possible to establish which one is finer, then we say that the two topologies are* comparable*.*

**Example 1.1.17.**
*The* cofinite topology $\tau_c$ *on* $\mathbb{R}$*, i.e.* $\tau_c := \{U \subseteq \mathbb{R} : U = \emptyset \text{ or } \mathbb{R} \setminus U \text{ is finite}\}$*, and the topology* $\tau_i$ *having* $\{(-\infty, a) : a \in \mathbb{R}\}$ *as a basis are incomparable. In fact, it is easy to see that* $\tau_i = \{(-\infty, a) : a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$ *as these are the unions of sets in the given basis. In particular, we have that* $\mathbb{R} - \{0\}$ *is in* $\tau_c$ *but not* $\tau_i$*. Moreover, we have that* $(-\infty, 0)$ *is in* $\tau_i$ *but not* $\tau_c$*. Hence,* $\tau_c$ *and* $\tau_i$ *are incomparable.*

It is always possible to construct at least two topologies on every set $X$ by choosing the collection of open sets to be as large or as small as possible:

- the *trivial topology*: every point of $X$ has only one neighbourhood which is $X$ itself. Equivalently, the only open subsets are $\emptyset$ and $X$. The only possible basis for the trivial topology is $\{X\}$.
- the *discrete topology*: given any point $x \in X$, every subset of $X$ containing $x$ is a neighbourhood of $x$. Equivalently, every subset of $X$ is open (actually clopen). In particular, the singleton $\{x\}$ is a neighbourhood of $x$ and actually is a basis of neighbourhoods of $x$. The collection of all singletons is a basis for the discrete topology.

The discrete topology on a set $X$ is finer than any other topology on $X$, while the trivial topology is coarser than all the others. Topologies on a set form thus a partially ordered set, having a maximal and a minimal element, respectively the discrete and the trivial topology.

A useful criterion to compare topologies on the same set is the following:

**Theorem 1.1.18** (Hausdorff's criterion)**.**
*Let* $\tau$ *and* $\tau'$ *two topologies on the same set* $X$*. For each* $x \in X$*, let* $\mathcal{B}(x)$ *a basis of neighbourhoods of* $x$ *in* $(X, \tau)$ *and* $\mathcal{B}'(x)$ *a basis of neighbourhoods of* $x$ *in* $(X, \tau')$*. Then:* $\tau \subseteq \tau'$ *iff* $\forall x \in X$*,* $\forall U \in \mathcal{B}(x)\ \exists V \in \mathcal{B}'(x)$ *s.t.* $x \in V \subseteq U$*.*

The Hausdorff criterion could be paraphrased by saying that smaller neighborhoods make larger topologies. This is a very intuitive theorem, because the smaller the neighbourhoods are the easier it is for a set to contain neighbourhoods of all its points and so the more open sets there will be.

*Proof.*
$\Rightarrow$ Suppose $\tau \subseteq \tau'$. Fixed any $x \in X$, let $U \in \mathcal{B}(x)$. Then, since $U \in \mathcal{F}_\tau(x)$, there exists $O \in \tau$ s.t. $x \in O \subseteq U$. But $O \in \tau$ implies by our assumption that $O \in \tau'$, so $U \in \mathcal{F}_{\tau'}(x)$. Hence, by Definition 1.1.12 for $\mathcal{B}'(x)$, there exists $V \in \mathcal{B}'(x)$ s.t. $V \subseteq U$.

$\Leftarrow$ Let $W \in \tau$. Then $W \in \mathcal{F}_\tau(x)$ for all $x \in X$. Since $\mathcal{B}(x)$ is a basis of neighbourhoods of $x$ in $(X, \tau)$, for each $x \in W$ there exists $U \in \mathcal{B}(x)$ such that $x \in U \subseteq W$. This together with the assumption guarantees that there exists $V \in \mathcal{B}'(x)$ s.t. $x \in V \subseteq U \subseteq W$. Hence, $W \in \mathcal{F}_{\tau'}(x)$ and so, by Remark 1.1.11, we have $W \in \tau'$. □

### 1.1.3 Reminder of some simple topological concepts

**Definition 1.1.19.** *Given a topological space $(X, \tau)$ and a subset $S$ of $X$, the* subset or induced topology *on $S$ is defined by $\tau_S := \{S \cap U \mid U \in \tau\}$. That is, a subset of $S$ is open in the subset topology if and only if it is the intersection of $S$ with an open set in $(X, \tau)$. Alternatively, we can define $\tau_S$ as the coarsest topology on $S$ for which the inclusion map $\iota : S \hookrightarrow X$ is continuous.*

Note that $(S, \tau_s)$ is a topological space in its own.

**Definition 1.1.20.** *Given a collection of topological space $(X_i, \tau_i)$, where $i \in I$ ($I$ is an index set possibly uncountable), the* product topology *on the Cartesian product $X := \prod_{i \in I} X_i$ is defined in the following way: a set $U$ is open in $X$ iff it is an arbitrary union of sets of the form $\prod_{i \in I} U_i$, where each $U_i \in \tau_i$ and $U_i \neq X_i$ for only finitely many $i$. Alternatively, we can define the product topology to be the coarsest topology for which all the canonical projections $p_i : X \to X_i$ are continuous.*

**Definition 1.1.21.**
*Given a topological space $X$, we define:*

- *The* closure *of a subset $A \subseteq X$ is the smallest closed set containing $A$. It will be denoted by $\bar{A}$. Equivalently, $\bar{A}$ is the intersection of all closed subsets of $X$ containing $A$.*
- *The* interior *of a subset $A \subseteq X$ is the largest open set contained in $A$. It will be denoted by $\mathring{A}$. Equivalently, $\mathring{A}$ is the union of all open subsets of $X$ contained in $A$.*

**Proposition 1.1.22.** *Given a top. space $X$ and $A \subseteq X$, the following hold.*

- *A point $x$ is a* closure point *of $A$, i.e. $x \in \bar{A}$, if and only if each neighborhood of $x$ has a nonempty intersection with $A$.*
- *A point $x$ is an* interior point *of $A$, i.e. $x \in \mathring{A}$, if and only if there exists a neighborhood of $x$ which entirely lies in $A$.*
- *$A$ is closed in $X$ iff $A = \bar{A}$.*
- *$A$ is open in $X$ iff $A = \mathring{A}$.*

*Proof.* (Recap Sheet 1)

**Example 1.1.23.** *Let $\tau$ be the standard euclidean topology on $\mathbb{R}$. Consider $X := (\mathbb{R}, \tau)$ and $Y := \big((0,1], \tau_Y\big)$, where $\tau_Y$ is the topology induced by $\tau$ on $(0,1]$. The closure of $(0, \frac{1}{2})$ in $X$ is $[0, \frac{1}{2}]$, but its closure in $Y$ is $(0, \frac{1}{2}]$.*

**Definition 1.1.24.** *Let $A$ and $B$ be two subsets of the same topological space $X$. $A$ is* dense *in $B$ if $B \subseteq \bar{A}$. In particular, $A$ is said to be dense in $X$ (or* everywhere dense*) if $\bar{A} = X$.*

**Examples 1.1.25.**

- *Standard examples of sets everywhere dense in the real line $\mathbb{R}$ (with the euclidean topology) are the set of rational numbers $\mathbb{Q}$ and the one of irrational numbers $\mathbb{R} - \mathbb{Q}$.*
- *A set $X$ is equipped with the discrete topology if and only if the whole space $X$ is the only dense set in itself.*
  If $X$ is endowed with the discrete topology then every subset is equal to its own closure (because every subset is closed), so the closure of a proper subset is always proper. Conversely, if $X$ is endowed with a topology $\tau$ s.t. the only dense subset of $X$ is itself, then for every proper subset $A$ its closure $\bar{A}$ is also a proper subset of $X$. Let $y \in X$ be arbitrary. Then $X \setminus \{y\}$ is a proper subset of $X$ and so it has to be equal to its own closure. Hence, $\{y\}$ is open. Since $y$ is arbitrary, this means that $\tau$ is the discrete topology.
- *Every non-empty subset of a set $X$ equipped with the trivial topology is dense, and every topology for which every non-empty subset is dense must be trivial.*
  If $X$ has the trivial topology and $A$ is any non-empty subset of $X$, then the only closed subset of $X$ containing $A$ is $X$. Hence, $\bar{A} = X$, i.e. $A$ is dense in $X$. Conversely, if $X$ is endowed with a topology $\tau$ for which every non-empty subset is dense, then the only non-empty subset of $X$ which is closed is $X$ itself. Hence, $\emptyset$ and $X$ are the only closed subsets of $\tau$. This means that $X$ has the trivial topology.

**Proposition 1.1.26.** *Let $X$ be a topological space and $A \subset X$. $A$ is dense in $X$ if and only if every nonempty open set in $X$ contains a point of $A$.*

*Proof.* If $A$ is dense in $X$, then by definition $\bar{A} = X$. Let $O$ be any nonempty open subset in $X$. Then for any $x \in O$ we have that $x \in \bar{A}$ and $O \in \mathcal{F}(x)$. Therefore, by Proposition 1.1.22, we have that $O \cap A \neq \emptyset$. Conversely, let $x \in X$. By definition of neighbourhood, for any $U \in \mathcal{F}(x)$ there exists an open subset $O$ of $X$ s.t. $x \in O \subseteq U$. Then $U \cap A \neq \emptyset$ since $O$ contains a point of $A$ by our assumption. Hence, by Proposition 1.1.22, we get $x \in \bar{A}$ and so that $A$ is dense in $X$. □

**Definition 1.1.27.** *A topological space $X$ is said to be separable if there exists a countable dense subset of $X$.*

**Example 1.1.28.**

- $\mathbb{R}$ *with the euclidean topology is separable.*
- *The space $\mathcal{C}([0,1])$ of all continuous functions from $[0,1]$ to $\mathbb{R}$ endowed with the uniform topology*[2] *is separable, since by the Weirstrass approximation theorem $\overline{\mathbb{Q}[x]} = \mathcal{C}([0,1])$.*

Let us briefly consider now the notion of convergence.
First of all let us concern with filters. When do we say that a filter $\mathcal{F}$ on a topological space $X$ converges to a point $x \in X$? Intuitively, if $\mathcal{F}$ has to converge to $x$, then the elements of $\mathcal{F}$, which are subsets of $X$, have to get somehow "smaller and smaller" about $x$, and the points of these subsets need to get "nearer and nearer" to $x$. This can be made more precise by using neighborhoods of $x$: we want to formally express the fact that, however small a neighborhood of $x$ is, it should contain some subset of $X$ belonging to the filter $\mathcal{F}$ and, consequently, all the elements of $\mathcal{F}$ which are contained in that particular one. But in view of Axiom (F3), this means that the neighborhood of $x$ under consideration must itself belong to the filter $\mathcal{F}$, since it must contain some element of $\mathcal{F}$.

**Definition 1.1.29.** *Given a filter $\mathcal{F}$ in a topological space $X$, we say that it converges to a point $x \in X$ if every neighborhood of $x$ belongs to $\mathcal{F}$, in other words if $\mathcal{F}$ is finer than the filter of neighborhoods of $x$.*

We recall now the definition of convergence of a sequence to a point and we see how it easily connects to the previous definition.

**Definition 1.1.30.** *Given a sequence of points $\{x_n\}_{n\in\mathbb{N}}$ in a topological space $X$, we say that it converges to a point $x \in X$ if for any $U \in \mathcal{F}(x)$ there exists $N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N$.*

If we now consider the filter $\mathcal{F}_S$ associated to the sequence $S := \{x_n\}_{n\in\mathbb{N}}$, i.e. $\mathcal{F}_S := \{A \subset X : |S \setminus A| < \infty\}$, then it is easy to see that:

**Proposition 1.1.31.** *Given a sequence of points $S := \{x_n\}_{n\in\mathbb{N}}$ in a topological space $X$, $S$ converges to a point $x \in X$ if and only if the associated filter $\mathcal{F}_S$ converges to $x$.*

[2] The uniform topology on $\mathcal{C}([0,1])$ is the topology induced by the supremum norm $\|\cdot\|_\infty$, i.e. the topology on $\mathcal{C}([0,1])$ having as basis of neighbourhoods of any $f \in \mathcal{C}([0,1])$ the collection $\{B_\varepsilon(f) : \varepsilon \in \mathbb{R}^+\}$ where $B_\varepsilon(f) := \{g \in \mathcal{C}([0,1]) : \|g - f\|_\infty < \varepsilon\}$ and $\|h\|_\infty := \sup_{x\in[0,1]} |h(x)|, \forall h \in \mathcal{C}([0,1])$

*Proof.* Set for each $m \in \mathbb{N}$, set $S_m := \{x_n \in S : n \geq m\}$. By Definition 1.1.30, $S$ converges to $x$ iff $\forall U \in \mathcal{F}(x), \exists N \in \mathbb{N} : S_N \subseteq U$. As $\mathcal{B} := \{S_m : m \in \mathbb{N}\}$ is a basis for $\mathcal{F}_S$ (c.f. Examples 1.1.8-c), we have that $\forall U \in \mathcal{F}(x), U \in \mathcal{F}_S$, which is equivalent to say that $\mathcal{F}(x) \subseteq \mathcal{F}_S$. □

### 1.1.4 Mappings between topological spaces

**Definition 1.1.32.** *Let $(X, \tau_X)$ and $(Y, \tau_Y)$ be two topological spaces. A map $f : X \to Y$ is* continuous *if the preimage of any open set in $Y$ is open in $X$, i.e. $\forall U \in \tau_Y$, $f^{-1}(U) := \{x \in X : f(x) \in U\} \in \tau_X$. Equivalently, given any point $x \in X$ and any $V \in \mathcal{F}(f(x))$ in $Y$, the preimage $f^{-1}(V) \in \mathcal{F}(x)$ in $X$.*

**Examples 1.1.33.**

- *Let $(X, \tau_X)$ and $(Y, \tau_Y)$ be two topological spaces. Any constant map $f : X \to Y$ is continuous.*
  Suppose that $f(x) := y$ for all $x \in X$ and some $y \in Y$. Let $U \in \tau_Y$. If $y \in U$ then $f^{-1}(U) = X$ and if $y \notin U$ then $f^{-1}(U) = \emptyset$. Hence, in either case, $f^{-1}(U)$ is open in $\tau_X$.
- *Let $(X, \tau_X)$ and $(Y, \tau_Y)$ be two topological spaces. If $g : X \to Y$ is continuous, then the restriction of $g$ to any subset $S$ of $X$ is also continuous w.r.t. the subset topology induced on $S$ by the topology on $X$.*
- *Let $X$ be a set endowed with the discrete topology, $Y$ be a set endowed with the trivial topology and $Z$ be any topological space. Any maps $f : X \to Z$ and $g : Z \to Y$ are continuous.*

**Definition 1.1.34.** *Let $(X, \tau_X)$ and $(Y, \tau_Y)$ be two topological spaces. A mapping $f : X \to Y$ is* open *if the image of any open set in $X$ is open in $Y$, i.e. $\forall V \in \tau_X$, $f(V) := \{f(x) : x \in V\} \in \tau_Y$. In the same way, a* closed mapping *$f : X \to Y$ sends closed sets to closed sets.*

Note that a map may be open, closed, both, or neither of them. Moreover, open and closed maps are not necessarily continuous.

**Example 1.1.35.** *If $Y$ is endowed with the discrete topology (i.e. all subsets are open and closed) then every function $f : X \to Y$ is both open and closed (but not necessarily continuous). For example, if we take the standard euclidean topology on $\mathbb{R}$ and the discrete topology on $\mathbb{Z}$ then the floor function $\mathbb{R} \to \mathbb{Z}$ is open and closed, but not continuous. (Indeed, the preimage of the open set $\{0\}$ is $[0, 1) \subset \mathbb{R}$, which is not open in the standard euclidean topology).*

If a continuous map $f$ is one-to-one, $f^{-1}$ does not need to be continuous.

**Example 1.1.36.**
*Let us consider* $[0,1) \subset \mathbb{R}$ *and* $S^1 \subset \mathbb{R}^2$ *endowed with the subspace topologies given by the euclidean topology on* $\mathbb{R}$ *and on* $\mathbb{R}^2$*, respectively. The map*

$$\begin{array}{rcl} f:[0,1) & \to & S^1 \\ t & \mapsto & (\cos 2\pi t, \sin 2\pi t). \end{array}$$

*is bijective and continuous but* $f^{-1}$ *is not continuous, since there are open subsets of* $[0,1)$ *whose image under* $f$ *is not open in* $S^1$*. (For example,* $[0,\frac{1}{2})$ *is open in* $[0,1)$ *but* $f([0,\frac{1}{2}))$ *is not open in* $S^1$*.)*

**Definition 1.1.37.** *A one-to-one map* $f$ *from* $X$ *onto* $Y$ *is a* homeomorphism *if and only if* $f$ *and* $f^{-1}$ *are both continuous. Equivalently, iff* $f$ *and* $f^{-1}$ *are both open (closed). If such a mapping exists,* $X$ *and* $Y$ *are said to be two* homeomorphic topological spaces*.*

In other words an homeomorphism is a one-to-one mapping which sends every open (resp. closed) set of $X$ in an open (resp. closed) set of $Y$ and viceversa, i.e. an homeomorphism is both an open and closed map. Note that the homeomorphism gives an equivalence relation on the class of all topological spaces.

**Examples 1.1.38.** *In these examples we consider any subset of* $\mathbb{R}^n$ *endowed with the subset topology induced by the Euclidean topology on* $\mathbb{R}^n$*.*

1. *Any open interval of* $\mathbb{R}$ *is homeomorphic to any other open interval of* $\mathbb{R}$ *and also to* $\mathbb{R}$ *itself.*
2. *A circle and a square in* $\mathbb{R}^2$ *are homeomorphic.*
3. *The circle* $S^1$ *with a point removed is homeomorphic to* $\mathbb{R}$*.*

Let us consider now the case when a set $X$ carries two different topologies $\tau_1$ and $\tau_2$. Then the following two properties are equivalent:

- the identity $\iota$ of $X$ is continuous as a mapping from $(X,\tau_1)$ and $(X,\tau_2)$
- the topology $\tau_1$ is finer than the topology $\tau_2$.

Therefore, $\iota$ is a homeomorphism if and only if the two topologies coincide.

*Proof.* Suppose that $\iota$ is continuous. Let $U \in \tau_2$. Then $\iota^{-1}(U) = U \in \tau_1$, hence $U \in \tau_1$. Therefore, $\tau_2 \subseteq \tau_1$. Conversely, assume that $\tau_2 \subseteq \tau_1$ and take any $U \in \tau_2$. Then $U \in \tau_1$ and by definition of identity we know that $\iota^{-1}(U) = U$. Hence, $\iota^{-1}(U) \in \tau_1$ and therefore, $\iota$ is continuous. □

**Proposition 1.1.39.** *Continuous maps preserve the convergence of sequences. That is, if* $f: X \to Y$ *is a continuous map between two topological spaces* $(X,\tau_X)$ *and* $(Y,\tau_Y)$ *and if* $\{x_n\}_{n\in\mathbb{N}}$ *is a sequence of points in* $X$ *convergent to a point* $x \in X$ *then* $\{f(x_n)\}_{n\in\mathbb{N}}$ *converges to* $f(x) \in Y$*.*

*Proof.* Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of points in $X$ convergent to a point $x \in X$ and let $U \in \mathcal{F}(f(x))$ in $Y$. It is clear from Definition 1.1.32 and Definition 1.1.5 that $f^{-1}(U) \in \mathcal{F}(x)$. Since $\{x_n\}_{n\in\mathbb{N}}$ converges to $x$, there exists $N \in \mathbb{N}$ s.t. $x_n \in f^{-1}(U)$ for all $n \geq N$. Then $f(x_n) \in U$ for all $n \geq N$. Hence, $\{f(x_n)\}_{n\in\mathbb{N}}$ converges to $f(x)$. □