Proof. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of points in X convergent to a point $x \in X$ and let $U \in \mathcal{F}(f(x))$ in Y. It is clear from Definition 1.1.32 and Definition 1.1.5 that $f^{-1}(U) \in \mathcal{F}(x)$. Since $\{x_n\}_{n\in\mathbb{N}}$ converges to x, there exists $N \in \mathbb{N}$ s.t. $x_n \in f^{-1}(U)$ for all $n \geq N$. Then $f(x_n) \in U$ for all $n \geq N$. Hence, $\{f(x_n)\}_{n\in\mathbb{N}}$ converges to $f(x)$. □

1.1.5 Hausdorff spaces

Definition 1.1.40. *A topological space X is said to be* Hausdorff *(or* separated*) if any two distinct points of X have neighbourhoods without common points; or equivalently if:*
(T2) two distinct points always lie in disjoint open sets.

In literature, the Hausdorff space are often called *T2-spaces* and the axiom (T2) is said to be the *separation axiom.*

Proposition 1.1.41. *In a Hausdorff space the intersection of all closed neighbourhoods of a point contains the point alone. Hence, the singletons are closed.*

Proof. Let us fix a point $x \in X$, where X is a Hausdorff space. Denote by C the intersection of all closed neighbourhoods of x. Suppose that there exists $y \in C$ with $y \neq x$. By definition of Hausdorff space, there exist a neighbourhood $U(x)$ of x and a neighbourhood $V(y)$ of y s.t. $U(x) \cap V(y) = \emptyset$. Therefore, $y \notin \overline{U(x)}$ because otherwise any neighbourhood of y (in particular $V(y)$) should have non-empty intersection with $U(x)$. Hence, $y \notin C$. □

Examples 1.1.42.

1. *Any metric space*[3] *is Hausdorff.*
   Indeed, for any $x, y \in (X, d)$ with $x \neq y$ just choose $0 < \varepsilon < \frac{1}{2}d(x, y)$ and you get $B_\varepsilon(x) \cap B_\varepsilon(y) = \emptyset$.
2. *Any set endowed with the discrete topology is a Hausdorff space.*
   Indeed, any singleton is open in the discrete topology so for any two distinct point x, y we have that $\{x\}$ and $\{y\}$ are disjoint and open.
3. *The only Hausdorff topology on a finite set is the discrete topology.*
   Let X be a finite set endowed with a Hausdorff topology τ. As X is finite, any subset S of X is finite and so S is a finite union of singletons. But since (X, τ) is Hausdorff, the previous proposition implies that any singleton is closed. Hence, any subset S of X is closed and so the τ has to be the discrete topology.

[3] Any metric space (X, d) is a topological space, because we can equip it with the topology induced by the metric d, i.e. the topology having as basis of neighbourhoods of any $x \in X$ the collection $\{B_\varepsilon(x) : \varepsilon \in \mathbb{R}^+\}$ where $B_\varepsilon(x) := \{y \in X : d(y, x) < \varepsilon\}$.

4. An infinite set with the cofinite topology is not Hausdorff.
In fact, any two non-empty open subsets O_1, O_2 in the cofinite topology on X are complements of finite subsets. Therefore, their intersection $O_1 \cap O_2$ is a complement of a finite subset, but X is infinite and so $O_1 \cap O_2 \neq \emptyset$. Hence, X is not Hausdorff.

1.2 Linear mappings between vector spaces

The basic notions from linear algebra are assumed to be well-known and so they are not recalled here. However, we briefly give again the definition of vector space and fix some general terminology for linear mappings between vector spaces. In this section we are going to consider vector spaces over the field $\mathbb{K}$ of real or complex numbers which is given the usual euclidean topology defined by means of the modulus.

Definition 1.2.1. *A set X with the two mappings:*

$$\begin{array}{ccll} X \times X & \to & X & \\ (x,y) & \mapsto & x+y & \textit{vector addition} \end{array}$$

$$\begin{array}{ccll} \mathbb{K} \times X & \to & X & \\ (\lambda,x) & \mapsto & \lambda x & \textit{scalar multiplication} \end{array}$$

is a vector space *(or* linear space*) over $\mathbb{K}$ if the following axioms are satisfied:*

(L1) *1.* $(x+y)+z = x+(y+z)$, $\forall\, x,y,z \in X$ *(associativity of +)*
2. $x+y = y+x$, $\forall\, x,y \in X$ *(commutativity of +)*
3. $\exists\, o \in X$*:* $x+o = x$, $\forall\, x, \in X$ *(neutral element for +)*
4. $\forall\, x \in X$, $\exists!\, -x \in X$ *s.t.* $x+(-x) = o$ *(inverse element for +)*

(L2) *1.* $\lambda(\mu x) = (\lambda\mu)x$, $\forall\, x \in X$, $\forall\, \lambda,\mu \in \mathbb{K}$
(compatibility of scalar multiplication with field multiplication)
2. $1x = x\, \forall\, x \in X$ *(neutral element for scalar multiplication)*
3. $(\lambda+\mu)x = \lambda x + \mu x$, $\forall\, x \in X$, $\forall\, \lambda,\mu \in \mathbb{K}$
(distributivity of scalar multiplication with respect to field addition)
4. $\lambda(x+y) = \lambda x + \lambda y$, $\forall\, x,y \in X$, $\forall\, \lambda \in \mathbb{K}$
(distributivity of scalar multiplication wrt vector addition)

Definition 1.2.2.
Let X, Y be two vector space over $\mathbb{K}$. A mapping $f : X \to Y$ is called linear mapping *or* homomorphism *if f preserves the vector space structure, i.e.* $f(\lambda x + \mu y) = \lambda f(x) + \mu f(y)\, \forall\, x,y \in X$, $\forall\, \lambda,\mu \in \mathbb{K}$.

Definition 1.2.3.

- *A linear mapping from X to itself is called* endomorphism*.*
- *A one-to-one linear mapping is called* monomorphism*. If S is a subspace of X, the identity map is a monomorphism and it is called* embedding*.*
- *An onto (surjective) linear mapping is called* epimorphism*.*
- *A bijective (one-to-one and onto) linear mapping between two vector spaces X and Y over $\mathbb{K}$ is called* (algebraic) isomorphism*. If such a map exists, we say that X and Y are (algebraically) isomorphic $X \cong Y$.*
- *An isomorphism from X into itself is called* automorphism*.*

It is easy to prove that: A linear mapping is one-to-one (injective) if and only if $f(x) = 0$ implies $x = 0$.

Definition 1.2.4. *A linear mapping from $X \to \mathbb{K}$ is called* linear functional *or* linear form *on X. The set of all linear functionals on X is called* algebraic dual *and it is denoted by X^*.*

Note that the dual space of a finite dimensional vector space X is isomorphic to X.

Chapter 2

Topological Vector Spaces

2.1 Definition and properties of a topological vector space

In this section we are going to consider vector spaces over the field $\mathbb{K}$ of real or complex numbers which is given the usual euclidean topology defined by means of the modulus.

Definition 2.1.1. *A vector space X over $\mathbb{K}$ is called a* topological vector space *(t.v.s.) if X is provided with a topology τ which is compatible with the vector space structure of X, i.e. τ makes the vector space operations both continuous.*

More precisely, the condition in the definition of t.v.s. requires that:

$$\begin{array}{ccll} X \times X & \to & X & \\ (x,y) & \mapsto & x+y & \quad \textit{vector addition} \end{array}$$

$$\begin{array}{ccll} \mathbb{K} \times X & \to & X & \\ (\lambda,x) & \mapsto & \lambda x & \quad \textit{scalar multiplication} \end{array}$$

are both continuous when we endow X with the topology τ, $\mathbb{K}$ with the euclidean topology, $X \times X$ and $\mathbb{K} \times X$ with the correspondent product topologies.

Remark 2.1.2. *If (X,τ) is a t.v.s then it is clear from Definition 2.1.1 that $\sum_{k=1}^{N} \lambda_k^{(n)} x_k^{(n)} \to \sum_{k=1}^{N} \lambda_k x_k$ as $n \to \infty$ w.r.t. τ if for each $k = 1, \dots, N$ as $n \to \infty$ we have that $\lambda_k^{(n)} \to \lambda_k$ w.r.t. the euclidean topology on $\mathbb{K}$ and $x_k^{(n)} \to x_k$ w.r.t. τ.*

Let us discuss now some examples and counterexamples of t.v.s.

Examples 2.1.3.

a) Every vector space X over $\mathbb{K}$ endowed with the trivial topology is a t.v.s..

b) Every normed vector space endowed with the topology given by the metric induced by the norm is a t.v.s. (see Exercise Sheet 1).

c) There are also examples of spaces whose topology cannot be induced by a norm or a metric but that are t.v.s., e.g. the space of infinitely differentiable functions, the spaces of test functions and the spaces of distributions endowed with suitable topologies (which we will discuss in details later on).

In general, a metric vector space is not a t.v.s.. Indeed, there exist metrics for which both the vector space operations of sum and product by scalars are discontinuous (see Exercise Sheet 1 for an example).

Proposition 2.1.4. *Every vector space X over $\mathbb{K}$ endowed with the discrete topology is not a t.v.s. unless $X = \{o\}$.*

Proof. Assume that it is a t.v.s. and take $o \neq x \in X$. The sequence $\alpha_n = \frac{1}{n}$ in $\mathbb{K}$ converges to 0 in the euclidean topology. Therefore, since the scalar multiplication is continuous, $\alpha_n x \to o$ by Proposition 1.1.39, i.e. for any neighbourhood U of o in X there exists $m \in \mathbb{N}$ s.t. $\alpha_n x \in U$ for all $n \geq m$. In particular, we can take $U = \{o\}$ since it is itself open in the discrete topology. Hence, $\alpha_m x = o$, which implies that $x = o$ and so a contradiction. □

Definition 2.1.5. *Two t.v.s. X and Y over $\mathbb{K}$ are* (topologically) isomorphic *if there exists a vector space isomorphism $X \to Y$ which is at the same time a homeomorphism (i.e. bijective, linear, continuous and inverse continuous).*

In analogy to Definition 1.2.3, let us collect here the corresponding terminology for mappings between two t.v.s..

Definition 2.1.6. *Let X and Y be two t.v.s. on $\mathbb{K}$.*

- *A* topological homomorphism *f from X to Y is a continuous linear mapping which is also open, i.e. every open set in X is mapped to an open set in $f(X)$ (endowed with the subset topology induced by Y).*
- *A* topological monomorphism *from X to Y is an injective topological homomorphism.*
- *A* topological isomorphism *from X to Y is a bijective topological homomorphism.*
- *A* topological automorphism *of X is a topological isomorphism from X into itself.*

Proposition 2.1.7. *Given a t.v.s. X, we have that:*

1. *For any $x_0 \in X$, the mapping $x \mapsto x + x_0$ (*translation *by x_0) is a homeomorphism of X onto itself.*
2. *For any $0 \neq \lambda \in \mathbb{K}$, the mapping $x \mapsto \lambda x$ (*dilation *by λ) is a topological automorphism of X.*

Proof. Both mappings are continuous as X is a t.v.s.. Moreover, they are bijections by the vector space axioms and their inverses $x \mapsto x - x_0$ and $x \mapsto \frac{1}{\lambda}x$ are also continuous. Note that the second map is also linear so it is a topological automorphism. □

Proposition 2.1.7–1 shows that the topology of a t.v.s. is always a *translation invariant topology*, i.e. all translations are homeomorphisms. Note that the translation invariance of a topology τ on a vector space X is not sufficient to conclude (X, τ) is a t.v.s..

Example 2.1.8. *If a metric d on a vector space X is translation invariant, i.e. $d(x + z, y + z) = d(x, y)$ for all $x, y, z \in X$ (e.g. the metric induced by a norm), then the topology induced by the metric is translation invariant and the addition is always continuous. However, the multiplication by scalars does not need to be necessarily continuous (take d to be the discrete metric, then the topology generated by the metric is the discrete topology which is not compatible with the scalar multiplication see Proposition 2.1.4).*

The translation invariance of the topology of a t.v.s. means, roughly speaking, that a t.v.s. X topologically looks about any point as it does about any other point. More precisely:

Corollary 2.1.9. *The filter $\mathcal{F}(x)$ of neighbourhoods of x in a t.v.s. X coincides with the family of the sets $O+x$ for all $O \in \mathcal{F}(o)$, where $\mathcal{F}(o)$ is the filter of neighbourhoods of the origin o (i.e. neutral element of the vector addition).*

Proof. (Exercise Sheet 1) □

Thus the topology of a t.v.s. is completely determined by the filter of neighbourhoods of any of its points, in particular by the filter of neighbourhoods of the origin o or, more frequently, by a base of neighbourhoods of the origin o. Therefore, we need some criteria on a filter of a vector space X which ensures that it is the filter of neighbourhoods of the origin w.r.t. some topology compatible with the vector structure of X.

Theorem 2.1.10. *A filter $\mathcal{F}$ of a vector space X over $\mathbb{K}$ is the filter of neighbourhoods of the origin w.r.t. some topology compatible with the vector structure of X if and only if*

1. *The origin belongs to every set $U \in \mathcal{F}$*
2. *$\forall U \in \mathcal{F}$, $\exists V \in \mathcal{F}$ s.t. $V + V \subset U$*
3. *$\forall U \in \mathcal{F}$, $\forall \lambda \in \mathbb{K}$ with $\lambda \neq 0$ we have $\lambda U \in \mathcal{F}$*
4. *$\forall U \in \mathcal{F}$, U is absorbing.*
5. *$\forall U \in \mathcal{F}$, $\exists V \in \mathcal{F}$ balanced s.t. $V \subset U$.*

Before proving the theorem, let us fix some definitions and notations:

Definition 2.1.11. *Let U be a subset of a vector space X.*

1. *U is* absorbing *(or* radial*) if $\forall x \in X$ $\exists \rho > 0$ s.t. $\forall \lambda \in \mathbb{K}$ with $|\lambda| \leq \rho$ we have $\lambda x \in U$. Roughly speaking, we may say that a subset is absorbing if it can be made by dilation to swallow every point of the whole space.*
2. *U is* balanced *(or* circled*) if $\forall x \in U$, $\forall \lambda \in \mathbb{K}$ with $|\lambda| \leq 1$ we have $\lambda x \in U$. Note that the line segment joining any point x of a balanced set U to $-x$ lies in U.*

Clearly, o must belong to every absorbing or balanced set. The underlying field can make a substantial difference. For example, if we consider the closed interval $[-1, 1] \subset \mathbb{R}$ then this is a balanced subset of $\mathbb{C}$ as real vector space, but if we take $\mathbb{C}$ as complex vector space then it is not balanced. Indeed, if we take $i \in \mathbb{C}$ we get that $i1 = i \notin [-1, 1]$.

Examples 2.1.12.

a) *In a normed space the unit balls centered at the origin are absorbing and balanced.*

b) *The unit ball B centered at $(\frac{1}{2}, 0) \in \mathbb{R}^2$ is absorbing but not balanced in the real vector space $\mathbb{R}^2$.* Indeed, B is a neighbourhood of the origin and so by Theorem 2.1.10-4 is absorbing. However, B is not balanced because for example if we take $x = (1, 0) \in B$ and $\lambda = -1$ then $\lambda x \notin B$.

c) *In the real vector space $\mathbb{R}^2$ endowed with the euclidean topology, the subset in Figure 2.1 is absorbing and the one in Figure 2.2 is balanced.*

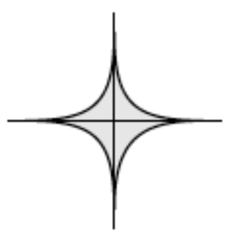

Figure 2.1: Absorbing

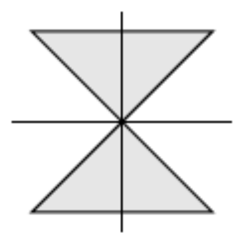

Figure 2.2: Balanced

d) *The polynomials $\mathbb{R}[x]$ are a balanced but not absorbing subset of the real space $\mathcal{C}([0, 1], \mathbb{R})$ of continuous real valued functions on $[0, 1]$.* Indeed, any multiple of a polynomial is still a polynomial but not every continuous function can be written as multiple of a polynomial.

e) *The subset* $A := \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| \leq |z_2|\}$ *of the complex space* $\mathbb{C}^2$ *endowed with the euclidean topology is balanced but* $\mathring{A}$ *is not balanced.* Indeed, $\forall\, (z_1, z_2) \in A$ and $\forall \lambda \in \mathbb{C}$ with $|\lambda| \leq 1$ we have that

$$|\lambda z_1| = |\lambda||z_1| \leq |\lambda||z_2| = |\lambda z_2|$$

i.e. $\lambda(z_1, z_2) \in A$. Hence, A is balanced. If we consider instead $\mathring{A} = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < |z_2|\}$ then $\forall\, (z_1, z_2) \in \mathring{A}$ and $\lambda = 0$ we have that $\lambda(z_1, z_2) = (0, 0) \notin \mathring{A}$. Hence, $\mathring{A}$ is not balanced.

Proposition 2.1.13.
a) If B *is a balanced subset of a t.v.s.* X *then so is* $\bar{B}$*.*
b) If B *is a balanced subset of a t.v.s.* X *and* $o \in \mathring{B}$ *then* $\mathring{B}$ *is balanced.*

Proof. (Exercise Sheet 1) □