Example 2.2.5. *Every vector space with an infinite number of elements endowed with the cofinite topology is not a t.v.s..* It is clear that in such topological space all singletons are closed, i.e. it is a (T1) space. Therefore, if it was a t.v.s. then by the previous results it should be a Hausdorff space which is not true as showed in Example 1.1.42.

2.3 Quotient topological vector spaces

Quotient topology

Let X be a topological space and $\sim$ be any equivalence relation on X. Then the *quotient set* $X/\sim$ is defined to be the set of all equivalence classes w.r.t. to $\sim$. The map $\phi : X \to X/\sim$ which assigns to each $x \in X$ its equivalence class $\phi(x)$ w.r.t. $\sim$ is called the *canonical map* or *quotient map*. Note that ϕ is surjective. We may define a topology on $X/\sim$ by setting that: a subset U of $X/\sim$ is open iff the preimage $\phi^{-1}(U)$ is open in X. This is called the *quotient topology* on $X/\sim$. Then it is easy to verify that:

- the quotient map ϕ is continuous.
- the quotient topology on $X/\sim$ is the finest topology on $X/\sim$ such that ϕ is continuous.

Note that the quotient map ϕ is not necessarily open or closed.

Example 2.3.1. *Consider $\mathbb{R}$ with the standard topology given by the modulus and define the following equivalence relation on $\mathbb{R}$:*

$$x \sim y \Leftrightarrow (x = y \vee \{x, y\} \subset \mathbb{Z}).$$

Let $\mathbb{R}/\sim$ be the quotient set w.r.t $\sim$ and $\phi : \mathbb{R} \to \mathbb{R}/\sim$ the correspondent quotient map. Let us consider the quotient topology on $\mathbb{R}/\sim$. Then ϕ is not an open map. In fact, if U is an open bounded subset of $\mathbb{R}$ containing an integer, then $\phi^{-1}(\phi(U)) = U \cup \mathbb{Z}$ which is not open in $\mathbb{R}$ with the standard topology. Hence, $\phi(U)$ is not open in $\mathbb{R}/\sim$ with the quotient topology.

For an example of quotient map which is not closed see Example 2.3.3 in the following.

Quotient vector space

Let X be a vector space and M a linear subspace of X. For two arbitrary elements $x, y \in X$, we define $x \sim_M y$ iff $x - y \in M$. It is easy to see that $\sim_M$ is an equivalence relation: it is reflexive, since $x - x = 0 \in M$ (every linear subspace contains the origin); it is symmetric, since $x - y \in M$ implies

$-(x-y) = y - x \in M$ (a linear subspace contains all scalar multiples of every of its elements); it is transitive, since $x - y \in M$, $y - z \in M$ implies $x - z = (x - y) + (y - z) \in M$ (when a linear subspace contains two vectors, it also contains their sum). Then X/M is defined to be the quotient set $X/\sim_M$, i.e. the set of all equivalence classes for the relation $\sim_M$ described above. The canonical (or quotient) map $\phi : X \to X/M$ which assigns to each $x \in X$ its equivalence class $\phi(x)$ w.r.t. the relation $\sim_M$ is clearly surjective. Using the fact that M is a linear subspace of X, it is easy to check that:

1. if $x \sim_M y$, then $\forall\, \lambda \in \mathbb{K}$ we have $\lambda x \sim_M \lambda y$.
2. if $x \sim_M y$, then $\forall\, z \in X$ we have $x + z \sim_M y + z$.

These two properties guarantee that the following operations are well-defined on X/M:

- vector addition: $\forall\, \phi(x), \phi(y) \in X/M$, $\phi(x) + \phi(y) := \phi(x + y)$
- scalar multiplication: $\forall\, \lambda \in \mathbb{K}$, $\forall\, \phi(x) \in X/M$, $\lambda\phi(x) := \phi(\lambda x)$

X/M with the two operations defined above is a vector space and therefore it is often called *quotient vector space*. Then the quotient map ϕ is clearly linear.

Quotient topological vector space

Let X be now a t.v.s. and M a linear subspace of X. Consider the quotient vector space X/M and the quotient map $\phi : X \to X/M$ defined in Section 2.3. Since X is a t.v.s, it is in particular a topological space, so we can consider on X/M the quotient topology defined in Section 2.3. We already know that in this topological setting ϕ is continuous but actually the structure of t.v.s. on X guarantees also that it is open.

Proposition 2.3.2. *For a linear subspace M of a t.v.s. X, the quotient mapping $\phi : X \to X/M$ is open (i.e. carries open sets in X to open sets in X/M) when X/M is endowed with the quotient topology.*

Proof. Let V open in X. Then we have

$$\phi^{-1}(\phi(V)) = V + M = \bigcup_{m \in M} (V + m)$$

Since X is a t.v.s, its topology is translation invariant and so $V + m$ is open for any $m \in M$. Hence, $\phi^{-1}(\phi(V))$ is open in X as union of open sets. By definition, this means that $\phi(V)$ is open in X/M endowed with the quotient topology. $\square$

It is then clear that ϕ carries neighborhoods of a point in X into neighborhoods of a point in X/M and viceversa. Hence, the neighborhoods of the origin in X/M are direct images under ϕ of the neighborhoods of the origin in X. In conclusion, when X is a t.v.s and M is a subspace of X, we can rewrite the definition of quotient topology on X/M in terms of neighborhoods as follows: *the filter of neighborhoods of the origin of X/M is exactly the image under ϕ of the filter of neighborhoods of the origin in X.*

It is not true, in general (not even when X is a t.v.s. and M is a linear subspace of X), that the quotient map is closed.

Example 2.3.3.
Consider $\mathbb{R}^2$ with the euclidean topology and the hyperbola $H := \{(x, y) \in \mathbb{R}^2 : xy = 1\}$. If M is one of the coordinate axes, then $\mathbb{R}^2/M$ can be identified with the other coordinate axis and the quotient map ϕ with the orthogonal projection on it. All these identifications are also valid for the topologies. The hyperbola H is closed in $\mathbb{R}^2$ but its image under ϕ is the complement of the origin on a straight line which is open.

Corollary 2.3.4. *For a linear subspace M of a t.v.s. X, the quotient space X/M endowed with the quotient topology is a t.v.s..*

Proof. For convenience, we denote here by A the vector addition in X/M and just by $+$ the vector addition in X. Let W be a neighbourhood of the origin o in X/M. We aim to prove that $A^{-1}(W)$ is a neighbourhood of (o, o) in $X/M \times X/M$. The continuity of the quotient map $\phi : X \to X/M$ implies that $\phi^{-1}(W)$ is a neighbourhood of the origin in X. Then, by Theorem 2.1.10-2 (we can apply the theorem because X is a t.v.s.), there exists V neighbourhood of the origin in X s.t. $V + V \subseteq \phi^{-1}(W)$. Hence, by the linearity and the surjectivity of ϕ, we get $A(\phi(V) \times \phi(V)) = \phi(V + V) \subseteq W$, i.e. $\phi(V) \times \phi(V) \subseteq A^{-1}(W)$. Since ϕ is also open, $\phi(V)$ is a neighbourhood of the origin o in X/M and so $A^{-1}(W)$ is a neighbourhood of (o, o) in $X/M \times X/M$.

A similar argument gives the continuity of the scalar multiplication. □

Proposition 2.3.5. *Let X be a t.v.s. and M a linear subspace of X. Consider X/M endowed with the quotient topology. Then the two following properties are equivalent:*
a) M is closed
b) X/M is Hausdorff

Proof. Recall that, since X is a t.v.s., Corollary 2.3.4 ensures that X/M endowed with the quotient topology is also a t.v.s..

Suppose that (a) holds. Then the complement U of M in X is open in X and so $\phi(U)$ is open in X/M since ϕ is an open map. Moreover, $\phi(U)$ does not contain the origin $o \in X/M$, because otherwise there would exist $y \in U$ such that $\phi(y) = o \in X/M$, i.e. $y \in M$ which is a contradiction. Hence, we have that $\phi(U)$ is a neighbourhood of each of point in X/M except the origin, i.e. (2.1) holds for X/M endowed with the quotient topology, which is equivalent to (b) by Proposition 2.2.3.

Conversely, suppose that (b) holds. By Corollary 2.2.4, this is equivalent to say that the set $\{o\} \subset X/M$ is closed w.r.t. the quotient topology. By the continuity of ϕ, we then have that $M = \phi^{-1}(\{o\})$ is closed in X, i.e. (a) holds. □

Corollary 2.3.6. *If X is a t.v.s., then $X/\overline{\{o\}}$ endowed with the quotient topology is a Hausdorff t.v.s.. $X/\overline{\{o\}}$ is said to be the* Hausdorff t.v.s. associated *with the t.v.s. X. When a t.v.s. X is Hausdorff, X and $X/\overline{\{o\}}$ are topologically isomorphic.*

Proof.
Since X is a t.v.s. and $\{o\}$ is a linear subspace of X, $\overline{\{o\}}$ is a closed linear subspace of X by Proposition 2.1.15-2. Then, by Corollary 2.3.4 and Proposition 2.3.5, $X/\overline{\{o\}}$ is a Hausdorff t.v.s.. If in addition X is Hausdorff, then Corollary 2.2.4 guarantees that $\overline{\{o\}} = \{o\}$ in X. Therefore, the quotient map $\phi : X \to X/\overline{\{o\}}$ is also injective because in this case $Ker(\phi) = \{o\}$. Hence, ϕ is a topological isomorphism (i.e. bijective, continuous, open, linear) between X and $X/\overline{\{o\}}$ which is indeed $X/\{o\}$. □

2.4 Continuous linear mappings between t.v.s.

Let X and Y be two vector spaces over $\mathbb{K}$ and $f : X \to Y$ a linear map. We define the *image* of f, and denote it by $Im(f)$, as the subset of Y:

$$Im(f) := \{y \in Y \,:\, \exists\, x \in X \text{ s.t. } y = f(x)\}.$$

We define the *kernel* of f, and denote it by $Ker(f)$, as the subset of X:

$$Ker(f) := \{x \in X \,:\, f(x) = 0\}.$$

Both $Im(f)$ and $Ker(f)$ are linear subspaces of Y and X, respectively. We have then the diagram:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\phi"] & Im(f) \arrow[r, "i"] & Y \\
X/Ker(f) \arrow[ur, "\bar{f}"] & &
\end{tikzcd}$$

where i is the natural injection of $Im(f)$ into Y, i.e. the mapping which to each element y of $Im(f)$ assigns that same element y regarded as an element of Y; ϕ is the canonical map of X onto its quotient $X/Ker(f)$. The mapping $\bar{f}$ is defined so as to make the diagram commutative, which means that:

$$\forall\, x \in X,\ f(x) = \bar{f}(\phi(x)).$$

Note that

- $\bar{f}$ is well-defined and injective.
  Indeed, $\phi(x) = \phi(y)$ if and only if $x - y \in Ker(f)$, i.e. $f(x-y) = 0$, which means by the linearity of f that $f(x) = f(y)$ and so by definition of $\bar{f}$ that $\bar{f}(\phi(x)) = \bar{f}(\phi(y))$.
- $\bar{f}$ is linear.
  This is an immediate consequence of the linearity of f and of the linear structure of $X/Ker(f)$.
- $\bar{f}$ is a surjective map from $X/Ker(f)$ onto $Im(f)$.
  The surjectivity is evident from the definition of $Im(f)$ and of $\bar{f}$.

The set of all linear maps (continuous or not) of a vector space X into another vector space Y is denoted by $\mathcal{L}(X;Y)$. Note that $\mathcal{L}(X;Y)$ is a vector space for the natural addition and multiplication by scalars of functions. Recall that when $Y = \mathbb{K}$, the space $\mathcal{L}(X;Y)$ is denoted by X^* and it is called the *algebraic dual* of X (see Definition 1.2.4).

Let us not turn to consider linear mappings between two t.v.s. X and Y. Since they posses a topological structure, it is natural to study in this setting continuous linear mappings.

Lemma 2.4.1. *Let $f : X \to Y$ a linear map between two t.v.s. X and Y. If Y is Hausdorff and f is continuous, then $Ker(f)$ is closed in X.*

Proof.
Clearly, $Ker(f) = f^{-1}(\{o\})$. Since Y is a Hausdorff t.v.s., $\{o\}$ is closed in Y and so, by the continuity of f, $Ker(f)$ is also closed in Y. $\square$

Note that $Ker(f)$ might be closed in X also when Y is not Hausdorff. For instance, when $f \equiv 0$ or when f is injective and X is Hausdorff.

Proposition 2.4.2. *Let $f : X \to Y$ be a linear map between two t.v.s. X and Y and consider $X/Ker(f)$ endowed with the quotient topology. Then f is continuous if and only if $\bar{f}$ is continuous.*

Proof.
Suppose f continuous and let U be an open subset in $Im(f)$ (endowed with the subspace topology induced by the topology on Y). Then $f^{-1}(U)$ is open in X. Since the quotient map $\phi : X \to X/Ker(f)$ is open, $\phi(f^{-1}(U))$ is open in $X/Ker(f)$. Also, by definition of $\bar{f}$, we have that $\bar{f}^{-1}(U) = \phi(f^{-1}(U))$. Hence, $\bar{f}^{-1}(U)$ is open in $X/Ker(f)$ and so the map $\bar{f}$ is continuous. Viceversa, suppose that $\bar{f}$ is continuous. Since $f = \bar{f} \circ \phi$ and ϕ is continuous, f is also continuous as composition of continuous maps. □

In general, the inverse of $\bar{f}$, which is well defined on $Im(f)$ since $\bar{f}$ is injective, is not continuous. In other words, $\bar{f}$ is not necessarily bi-continuous.

The set of all continuous linear maps of a t.v.s. X into another t.v.s. Y is denoted by $L(X;Y)$ and it is a vector subspace of $\mathcal{L}(X;Y)$. When $Y = \mathbb{K}$, the space $L(X;Y)$ is usually denoted by X' which is called the *topological dual* of X, in order to underline the difference with X^* the algebraic dual of X. X' is a vector subspace of X^* and is exactly the vector space of all continuous linear functionals, or continuous linear forms, on X. The vector spaces X' and $L(X;Y)$ will play an important role in the forthcoming and will be equipped with various topologies.

2.5 Completeness for t.v.s.

This section aims to treat completeness for most general types of topological vector spaces, beyond the traditional metric framework. As well as in the case of metric spaces, we need to introduce the definition of a Cauchy sequence in a t.v.s..

Definition 2.5.1. *A sequence $S := \{x_n\}_{n\in\mathbb{N}}$ of points in a t.v.s. X is said to be a* Cauchy sequence *if*

$$\forall\, U \in \mathcal{F}(o)\, in\, X,\ \exists\, N \in \mathbb{N} :\ x_m - x_n \in U,\ \forall m, n \geq N. \tag{2.2}$$

This definition agrees with the usual one if the topology of X is defined by a translation-invariant metric d. Indeed, in this case, a basis of neighbourhoods of the origin is given by all the open balls centered at the origin. Therefore, $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in such (X,d) iff $\forall\, \varepsilon > 0$, $\exists\, N \in \mathbb{N}$: $x_m - x_n \in B_\varepsilon(o)$, $\forall m, n \geq N$, i.e. $d(x_m, x_n) = d(x_m - x_n, o) < \varepsilon$.

By using the subsequences $S_m := \{x_n \in S \,:\, n \geq m\}$ of S, we can easily rewrite (2.2) in the following way

$$\forall\, U \in \mathcal{F}(o) \,\text{in}\, X,\ \exists\, N \in \mathbb{N} :\ S_N - S_N \subset U.$$

As we have already observed in Chapter 1, the collection $\mathcal{B} := \{S_m \,:\, m \in \mathbb{N}\}$ is a basis of the filter $\mathcal{F}_S$ associated with the sequence S. This immediately suggests what the definition of a Cauchy filter should be:

Definition 2.5.2. *A filter $\mathcal{F}$ on a subset A of a t.v.s. X is said to be a* Cauchy filter *if*

$$\forall\, U \in \mathcal{F}(o) \,\textit{in}\, X,\ \exists\, M \subset A :\ M \in \mathcal{F} \textit{ and } M - M \subset U.$$