Theorem 2.5.12.
Let X be a Hausdorff t.v.s.. Then there exists a complete Hausdorff t.v.s. $\hat{X}$ and a mapping $i : X \to \hat{X}$ with the following properties:
a) The mapping i is a topological monomorphism.
b) The image of X under i is dense in $\hat{X}$.
c) For every complete Hausdorff t.v.s. Y and for every continuous linear map $f : X \to Y$, there is a continuous linear map $\hat{f} : \hat{X} \to Y$ such that the following diagram is commutative:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle i} & \nearrow{\scriptstyle \hat{f}} & \\ \hat{X} & & \end{array}$$

Furthermore:
I) Any other pair $(\hat{X}_1, i_1)$, consisting of a complete Hausdorff t.v.s. $\hat{X}_1$ and of a mapping $i_1 : X \to \hat{X}_1$ such that properties (a) and (b) hold substituting $\hat{X}$ with $\hat{X}_1$ and i with i_1, is topologically isomorphic to $(\hat{X}, i)$. This means that there is a topological isomorphism j of $\hat{X}$ onto $\hat{X}_1$ such that the following diagram is commutative:

$$\begin{array}{ccc} X & \xrightarrow{i_1} & \hat{X}_1 \\ \downarrow{\scriptstyle i} & \nearrow{\scriptstyle j} & \\ \hat{X} & & \end{array}$$

II) Given Y and f as in property (c), the continuous linear map $\hat{f}$ is unique.

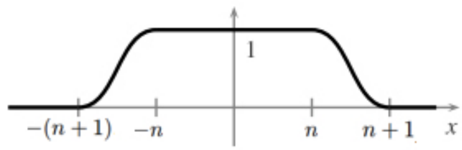
Let us use the previous theorem to compute the completion of a very well-known infinite dimensional Hausdorff t.v.s..

Example 2.5.13. *Let $\mathcal{C}(\mathbb{R})$ be the vector space of real valued functions defined and continuous on the real line and $\mathcal{C}_c(\mathbb{R})$ the linear subspace of all functions $f \in \mathcal{C}(\mathbb{R})$ whose support $\mathrm{supp}(f)$ is a compact subset of $\mathbb{R}$. We endow $\mathcal{C}(\mathbb{R})$ with the topology τ having $\{N_{\varepsilon,n} : \varepsilon \in \mathbb{R}^+, n \in \mathbb{N}\}$ as a basis of neighbourhoods of the origin where $N_{\varepsilon,n} := \left\{ f \in \mathcal{C}(\mathbb{R}) : \sup_{|t|\leq n} |f(t)| \leq \varepsilon \right\}$. It is possible to show that $(\mathcal{C}(\mathbb{R}), \tau)$ is a Hausdorff complete t.v.s. (see Exercise Sheet 3). Let τ_{sub} be the subspace topology induced by τ on $\mathcal{C}_c(\mathbb{R})$.*
Claim: The completion of $(\mathcal{C}_c(\mathbb{R}), \tau_{sub})$ is topologically isomorphic to $(\mathcal{C}(\mathbb{R}), \tau)$.

Proof. of Claim.
Since $(\mathcal{C}(\mathbb{R}), \tau)$ is a Hausdorff t.v.s., $(\mathcal{C}_c(\mathbb{R}), \tau_{sub})$ is also a Hausdorff t.v.s. and the inclusion map $i_c : \mathcal{C}_c(\mathbb{R}) \to \mathcal{C}(\mathbb{R})$ is clearly injective, linear, continuous

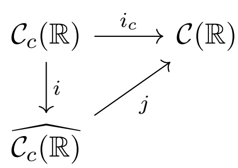
and open, i.e. a topological monomorphism. We first show that $(\mathcal{C}_c(\mathbb{R}), \tau_{sub})$ is dense in $(\mathcal{C}(\mathbb{R}), \tau)$.

Let $f \in \mathcal{C}(\mathbb{R})$, $\varepsilon \in \mathbb{R}^+$ and $n \in \mathbb{N}$. Take $g \in \mathcal{C}(\mathbb{R})$ such that $g(x) = 1$ for all $x \in [-n, n]$ and $\operatorname{supp}(g) \subseteq [-(n+1), n+1]$.



Define $h(x) := f(x)g(x)$ for all $x \in \mathbb{R}$. Then h is continuous as product of continuous functions and $\operatorname{supp}(h) \subseteq \operatorname{supp}(f) \cap \operatorname{supp}(g) \subseteq [-(n+1), n+1]$, i.e. $h \in \mathcal{C}_c(\mathbb{R})$. Also, $h - f \in N_{\varepsilon,n}$ since $\sup_{|t|\leq n} |h(t) - f(t)| = \sup_{|t|\leq n} |f(t) - f(t)| = 0 < \varepsilon$. Hence, $h \in (f + N_{\varepsilon,n}) \cap \mathcal{C}_c(\mathbb{R})$, that is, $(f + N_{\varepsilon,n}) \cap \mathcal{C}_c(\mathbb{R}) \neq \emptyset$, which completes the proof of the density.

We have therefore showed that $(\mathcal{C}_c(\mathbb{R}), \tau_{sub})$ together with i_c fulfill both the property (a) and (b) of Theorem 2.5.12. Then by Part (I) of Theorem 2.5.12 we have that there exists a topological isomorphism j of $\widehat{\mathcal{C}_c(\mathbb{R})}$ onto $\mathcal{C}(\mathbb{R})$ such that the following diagram is commutative:



which completes the proof of our claim. $\square$

Chapter 3

Finite dimensional topological vector spaces

3.1 Finite dimensional Hausdorff t.v.s.

Let X be a vector space over the field $\mathbb{K}$ of real or complex numbers. We know from linear algebra that the (algebraic) dimension of X, denoted by $\dim(X)$, is the cardinality of a basis of X. If $\dim(X)$ is finite, we say that X is *finite dimensional* otherwise X is *infinite dimensional*. In this section we are going to focus on finite dimensional vector spaces.

Let $\{e_1, \ldots, e_d\}$ be a basis of X, i.e. $\dim(X) = d$. Given any vector $x \in X$ there exist unique $x_1, \ldots, x_d \in \mathbb{K}$ s.t. $x = x_1 e_1 + \cdots + x_d e_d$. This can be precisely expressed by saying that the mapping

$$\begin{array}{ccc} \mathbb{K}^d & \to & X \\ (x_1, \ldots, x_d) & \mapsto & x_1 e_1 + \cdots + x_d e_d \end{array}$$

is an algebraic isomorphism (i.e. linear and bijective) between X and $\mathbb{K}^d$. In other words: *If X is a finite dimensional vector space then X is algebraically isomorphic to $\mathbb{K}^{\dim(X)}$.*

If now we give to X the t.v.s. structure and we consider $\mathbb{K}$ endowed with the euclidean topology, then it is natural to ask if such an algebraic isomorphism is by any chance a topological one, i.e. if it preserves the t.v.s. structure. The following theorem shows that if X is a finite dimensional Hausdorff t.v.s. then the answer is yes: X is topologically isomorphic to $\mathbb{K}^{\dim(X)}$.

It is worth to observe that usually in applications we deal always with Hausdorff t.v.s., therefore it makes sense to mainly focus on them.

Theorem 3.1.1. *Let X be a finite dimensional Hausdorff t.v.s. over $\mathbb{K}$ (where $\mathbb{K}$ is endowed with the euclidean topology). Then:*
a) X is topologically isomorphic to $\mathbb{K}^d$, where $d = \dim(X)$.
b) Every linear functional on X is continuous.
c) Every linear map of X into any t.v.s. Y is continuous.

Before proving the theorem let us recall some lemmas about the continuity of linear functionals on t.v.s..

Lemma 3.1.2.
Let X be a t.v.s over $\mathbb{K}$ and $v \in X$. Then the following mapping is continuous.

$$\begin{array}{rccc} \varphi_v : & \mathbb{K} & \to & X \\ & \xi & \mapsto & \xi v. \end{array}$$

Proof. For any $\xi \in \mathbb{K}$, we have $\varphi_v(\xi) = M(\psi_v(\xi))$, where $\psi_v : \mathbb{K} \to \mathbb{K} \times X$ given by $\psi_v(\xi) := (\xi, v)$ is clearly continuous by definition of product topology and $M : \mathbb{K} \times X \to X$ is the scalar multiplication in the t.v.s. X which is continuous by definition of t.v.s.. Hence, φ_v is continuous as composition of continuous mappings. □

Lemma 3.1.3. *Let X be a t.v.s. over $\mathbb{K}$ and L a linear functional on X. Assume $L(x) \neq 0$ for some $x \in X$. Then the following are equivalent:*
a) L is continuous.
b) The null space $Ker(L)$ is closed in X
c) $Ker(L)$ is not dense in X.
d) L is bounded in some neighbourhood of the origin in X, i.e. $\exists\ V \in \mathcal{F}(o)$ s.t. $\sup_{x\in V} |L(x)| < \infty$.

Proof. (see Exercise Sheet 3)

Proof. of Theorem 3.1.1
Let $\{e_1, \ldots, e_d\}$ be a basis of X and let us consider the mapping

$$\begin{array}{rccc} \varphi : & \mathbb{K}^d & \to & X \\ & (x_1, \ldots, x_d) & \mapsto & x_1 e_1 + \cdots + x_d e_d. \end{array}$$

As noted above, this is an algebraic isomorphism. Therefore, to conclude a) it remains to prove that φ is also a homeomorphism.

Step 1: φ is continuous.
When $d = 1$, we simply have $\varphi \equiv \varphi_{e_1}$ and so we are done by Lemma 3.1.2. When $d > 1$, for any $(x_1, \ldots, x_d) \in \mathbb{K}^d$ we can write: $\varphi(x_1, \ldots, x_d) = A(\varphi_{e_1}(x_1), \ldots, \varphi_{e_d}(x_d)) = A((\varphi_{e_1} \times \cdots \times \varphi_{e_d})(x_1, \ldots, x_d))$ where each φ_{e_j}

is defined as above and $A : X \times X \to X$ is the vector addition in the t.v.s. X. Hence, φ is continuous as composition of continuous mappings.

Step 2: φ is open and b) holds.
We prove this step by induction on the dimension $\dim(X)$ of X.
For $\dim(X) = 1$, it is easy to see that φ is open, i.e. that the inverse of φ:

$$\begin{array}{cccc} \varphi^{-1}: & X & \to & \mathbb{K} \\ & x = \xi e_1 & \mapsto & \xi \end{array}$$

is continuous. Indeed, we have that

$$Ker(\varphi^{-1}) = \{x \in X : \varphi^{-1}(x) = 0\} = \{\xi e_1 \in X : \xi = 0\} = \{o\},$$

which is closed in X, since X is Hausdorff. Hence, by Lemma 3.1.3, φ^{-1} is continuous. This implies that b) holds. In fact, if L is a non-identically zero functional on X (when $L \equiv 0$, there is nothing to prove), then there exists a $o \neq \tilde{x} \in X$ s.t. $L(\tilde{x}) \neq 0$. W.l.o.g. we can assume $L(\tilde{x}) = 1$. Now for any $x \in X$, since $\dim(X) = 1$, we have that $x = \xi\tilde{x}$ for some $\xi \in \mathbb{K}$ and so $L(x) = \xi L(\tilde{x}) = \xi$. Hence, $L \equiv \varphi^{-1}$ which we proved to be continuous.

Let $d \in \mathbb{N}$ with $d > 1$ and suppose now that both φ^{-1} is continuous and b) holds for $\dim(X) \leq d-1$. Let us first show that b) holds when $\dim(X) = d$. Let L be a non-identically zero functional on X (when $L \equiv 0$, there is nothing to prove), then there exists a $o \neq \tilde{x} \in X$ s.t. $L(\tilde{x}) \neq 0$. W.l.o.g. we can assume $L(\tilde{x}) = 1$. Note that for any $x \in X$ the element $x - \tilde{x}L(x) \in Ker(L)$. Therefore, if we take the canonical mapping $\phi : X \to X/Ker(L)$ then $\phi(x) = \phi(\tilde{x}L(x)) = L(x)\phi(\tilde{x})$ for any $x \in X$. This means that $X/Ker(L) = span\{\phi(\tilde{x})\}$ i.e. $\dim(X/Ker(L)) = 1$. Hence, $\dim(Ker(L)) = d-1$ and so by inductive assumption $Ker(L)$ is topologically isomorphic to $\mathbb{K}^{d-1}$ [1] This implies that $Ker(L)$ is a complete subspace of X. Then, by Proposition 2.5.8-a), $Ker(L)$ is closed in X and so by Lemma 3.1.3 we get L is continuous. By induction, we can conclude that b) holds whatever is the dimension of X.

This immediately implies that φ^{-1} is continuous whatever is $dim(X)$ (and so that a) holds for any dimension). In fact, the mapping

$$\begin{array}{cccc} \varphi^{-1}: & X & \to & \mathbb{K}^d \\ & x = \sum_{j=1}^{d} x_j e_j & \mapsto & (x_1, \ldots, x_d) \end{array}$$

is continuous. Now for any $x = \sum_{j=1}^{d} x_j e_j \in X$ we can write $\varphi^{-1}(x) =$

[1] Note that we can apply the inductive assumption not only because $\dim(Ker(L)) = d-1$ but also because $Ker(L)$ is a Hausdorff t.v.s. since it is a linear subspace of X which is an Hausdorff t.v.s..

$(L_1(x), \dots, L_d(x))$, where for any $j \in \{1, \dots, d\}$ we define $L_j : X \to \mathbb{K}$ by $L_j(x) := x_j e_j$. Since b) holds for any dimension, we know that each L_j is continuous and so φ^{-1} is continuous.

Step 3: The statement c) holds.
Let $f : X \to Y$ be linear and $\{e_1, \dots, e_d\}$ be a basis of X. For any $j \in \{1, \dots, d\}$ we define $b_j := f(e_j) \in Y$. Hence, for any $x = \sum_{j=1}^d x_j e_j \in X$ we have $f(x) = f(\sum_{j=1}^d x_j e_j) = \sum_{j=1}^d x_j b_j$. We can rewrite f as composition of continuous maps i.e. $f(x) = A((\varphi_{b_1} \times \dots \times \varphi_{b_d})(\varphi^{-1}(x))$ where:

- φ^{-1} is continuous by a)
- each φ_{b_j} is continuous by Lemma 3.1.2
- A is the vector addition on X and so it is continuous since X is a t.v.s..

Hence, f is continuous. □

Corollary 3.1.4 (Tychonoff theorem). *Let $d \in \mathbb{N}$. The only topology that makes $\mathbb{K}^d$ a Hausdorff t.v.s. is the euclidean topology. Equivalently, on a finite dimensional vector space there is a unique topology that makes it into a Hausdorff t.v.s..*

Proof.
We already know that $\mathbb{K}^d$ endowed with the euclidean topology τ_e is a Hausdorff t.v.s. of dimension d. Let us consider another topology τ on $\mathbb{K}^d$ s.t. $(\mathbb{K}^d, \tau)$ is also Hausdorff t.v.s.. Then Theorem 3.1.1-a) ensures that the identity map between $(\mathbb{K}^d, \tau_e)$ and $(\mathbb{K}^d, \tau)$ is a topological isomorphism. Hence, as observed at the end of Section 1.1.4 p.10, we get that $\tau \equiv \tau_e$. □

Corollary 3.1.5. *Every finite dimensional Hausdorff t.v.s. is complete.*

Proof.
Let X be a Hausdorff t.v.s with $\dim(X) = d < \infty$. Then, by Theorem 3.1.1-a), X is topologically isomorphic to $\mathbb{K}^d$ endowed with the euclidean topology. Since the latter is a complete Hausdorff t.v.s., so is X. □

Corollary 3.1.6. *Every finite dimensional linear subspace of a Hausdorff t.v.s. is closed.*

Proof.
Let S be a linear subspace of a Hausdorff t.v.s. (X, τ) and assume that $\dim(S) = d < \infty$. Then S endowed with the subspace topology induced by τ is itself a Hausdorff t.v.s.. Hence, by Corollary 3.1.5 S is complete and therefore closed by Proposition 2.5.8-a). □

3.2 Connection between local compactness and finite dimensionality

Let $d \in \mathbb{N}$ and $\mathbb{K}^d$ be endowed with euclidean topology. By the Heine-Borel property (a subset of $\mathbb{K}^d$ is closed and bounded iff it is compact), $\mathbb{K}^d$ has a basis of compact neighbourhoods of the origin (i.e. the closed balls centered at the origin in $\mathbb{K}^d$). Thus, in virtue of Theorem 3.1.1, the origin (and consequently every point) of a finite dimensional Hausdorff t.v.s. has a basis of neighbourhoods consisting of compact subsets. This means that *a finite dimensional Hausdorff t.v.s. is always locally compact*. Actually also the converse is true and gives the following beautiful characterization of finite dimensional Hausdorff t.v.s due to F. Riesz.

Theorem 3.2.1. *A Hausdorff t.v.s. is locally compact if and only if it is finite dimensional.*

For convenience let us recall the notions of compactness and local compactness for topological spaces before proving the theorem.

Definition 3.2.2. *A topological space X is* compact *if every open covering of X contains a finite subcovering. i.e. for any arbitrary collection $\{U_i\}_{i\in I}$ of open subsets of X s.t. $X \subseteq \cup_{i\in I} U_i$ there exists a finite subset J of I s.t. $X \subseteq \cup_{i\in J} U_i$.*

Definition 3.2.3. *A topological space X is* locally compact *if for every point $x \in X$ there exists a basis of compact neighbourhoods of x.*

We also remind two typical properties of compact spaces.

Proposition 3.2.4.

a) *A closed subset of a compact space is compact.*
b) *Let f be a continuous mapping from a compact space X into a topological space Y. Then $f(X)$ is a compact subset of Y.*

Just a small side remark: every compact t.v.s. is also locally compact but there exist locally compact t.v.s. that are not compact such as: $\mathbb{K}^d$ with the euclidean topology.

Proof. of Theorem 3.2.1
As mentioned in the introduction of this section, if X is a finite dimensional Hausdorff t.v.s. then it is locally compact. Thus, we need to show only the converse.

Let X be a locally compact Hausdorff t.v.s., and K a compact neighborhood of o in X. As K is compact and as $\frac{1}{2}K$ is a neighborhood of the origin (see Theorem 2.1.10-3), there is a finite family of points $x_1, \dots, x_r \in X$ s.t.

$$K \subseteq \bigcup_{i=1}^{r} (x_i + \frac{1}{2}K).$$

Let $M := span\{x_1, \dots, x_r\}$. Then M is a finite dimensional linear subspace of X which is a Hausdorff t.v.s.. Hence, M is closed in X by Corollary 3.1.6. Therefore, the quotient space X/M is Hausdorff t.v.s. by Proposition 2.3.5.

Let $\phi : X \to X/M$ be the canonical mapping. As $K \subseteq M + \frac{1}{2}K$, we have $\phi(K) \subseteq \phi(M) + \phi(\frac{1}{2}K) = \frac{1}{2}\phi(K)$, i.e. $2\phi(K) \subseteq \phi(K)$. By iterating we get $\phi(2^n K) \subseteq \phi(K)$ for any $n \in \mathbb{N}$. As K is absorbing (see Theorem 2.1.10-5), we have $X = \bigcup_{n=1}^{\infty} 2^n K$. Thus

$$X/M = \phi(X) = \bigcup_{n=1}^{\infty} \phi(2^n K) \subseteq \phi(K).$$

Since ϕ is continuous, Proposition 3.2.4-b) guarantees that $\phi(K)$ is compact. Thus X/M is compact. We claim that X/M must be of zero dimension, i.e. reduced to one point. This concludes the proof because it implies $\dim(X) = \dim(M) < \infty$.

Let us prove the claim by contradiction. Suppose $\dim(X/M) > 0$ then X/M contains a subset of the form $\mathbb{R}\bar{x}$ for some $\bar{o} \neq \bar{x} \in X/M$. Since such a subset is closed and X/M is compact, by Proposition 3.2.4-a), $\mathbb{R}\bar{x}$ is also compact which is a contradiction. □