Chapter 4

Locally convex topological vector spaces

4.1 Definition by neighbourhoods

Let us start this section by briefly recalling some basic properties of convex subsets of a vector space over $\mathbb{K}$ (where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$).

Definition 4.1.1. *A subset S of a vector space X over $\mathbb{K}$ is* convex *if, whenever S contains two points x and y, S also contains the segment of straight line joining them, i.e.*

$$\forall\, x, y \in S,\ \forall\, \alpha, \beta \in \mathbb{R} \text{ s.t. } \alpha, \beta \geq 0 \text{ and } \alpha + \beta = 1, \alpha x + \beta y \in S.$$

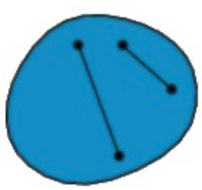

Figure 4.1: Convex set

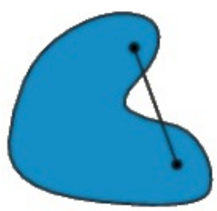

Figure 4.2: Not convex set

Examples 4.1.2.

a) *The convex subsets of $\mathbb{R}$ are simply the intervals of $\mathbb{R}$. Examples of convex subsets of $\mathbb{R}^2$ are simple regular polygons. The Platonic solids are convex subsets of $\mathbb{R}^3$. Hyperplanes and halfspaces in $\mathbb{R}^n$ are convex.*
b) *Balls in a normed space are convex.*
c) *Consider a topological space X and the set $\mathcal{C}(X)$ of all real valued functions defined and continuous on X. $\mathcal{C}(X)$ with the pointwise addition and scalar*

multiplication of functions is a vector space. Fixed $g \in \mathcal{C}(X)$, the subset $S := \{f \in \mathcal{C}(X) : f(x) \geq g(x), \forall x \in X\}$ is convex.

d) *Consider the vector space $\mathbb{R}[x]$ of all polynomials in one variable with real coefficients. Fixed $n \in \mathbb{N}$ and $c \in \mathbb{R} \setminus \{0\}$, the subset of all polynomials in $\mathbb{R}[x]$ such that the coefficient of the term of degree n is equal to c is convex.*

Proposition 4.1.3.
Let X be a vector space over $\mathbb{K}$. The following properties hold.

- *$\emptyset$ and X are convex.*
- *Arbitrary intersections of convex sets are convex sets.*
- *Unions of convex sets are generally not convex.*
- *The sum of two convex sets is convex.*
- *The image and the preimage of a convex set under a linear map is convex.*

Definition 4.1.4. *Let S be any subset of a vector space X over $\mathbb{K}$. We define the* convex hull *of S, denoted by $conv(S)$, to be the set of all finite convex linear combinations of elements of S, i.e.*

$$conv(S) := \left\{ \sum_{i=1}^{n} \lambda_i x_i : x_i \in S, \lambda_i \in [0,1], \sum_{i=1}^{n} \lambda_i = 1, n \in \mathbb{N} \right\}.$$

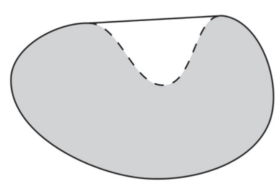

Figure 4.3: The solid line is the border of the convex hull of the shaded set

Proposition 4.1.5.
Let S, T be arbitrary subsets of a vector space X over $\mathbb{K}$. The following hold.

a) *$conv(S)$ is convex*
b) *$S \subseteq conv(S)$*
c) *A set is convex if and only if it is equal to its own convex hull.*
d) *If $S \subseteq T$ then $conv(S) \subseteq conv(T)$*
e) *$conv(conv(S)) = conv(S)$.*
f) *$conv(S + T) = conv(S) + conv(T)$.*
g) *The convex hull of S is the smallest convex set containing S, i.e. $conv(S)$ is the intersection of all convex sets containing S.*
h) *The convex hull of a balanced set is balanced*

Proof. (Christmas Assignment) □

Definition 4.1.6. *A subset S of a vector space X over $\mathbb{K}$ is* absolutely convex (abc) *if it is convex and balanced.*

Let us come back now to topological vector spaces.

Proposition 4.1.7. *The closure and the interior of convex sets in a t.v.s. are convex sets.*

Before proving it, let us recall that given a continuous map f between two topological spaces X and Y we have that $f(\overline{A}) \subseteq \overline{f(A)}$ for any $A \subseteq X$.

Proof. Let S be a convex subset of a t.v.s. X. For any $\lambda \in [0,1]$, we define:

$$\begin{array}{rccc} \varphi_\lambda : & X \times X & \to & X \\ & (x,y) & \mapsto & \lambda x + (1-\lambda)y \end{array}.$$

Note that each φ_λ is continuous by the continuity of addition and scalar multiplication in the t.v.s. X. Since S is convex, for any $\lambda \in [0,1]$ we have that $\varphi_\lambda(S \times S) \subseteq S$ and so $\overline{\varphi_\lambda(S \times S)} \subseteq \overline{S}$. The continuity of φ_λ guarantees that $\varphi_\lambda(\overline{S \times S}) \subseteq \overline{\varphi_\lambda(S \times S)}$. Hence, we can conclude that $\varphi_\lambda(\overline{S} \times \overline{S}) = \varphi_\lambda(\overline{S \times S}) \subseteq \overline{S}$, i.e. $\overline{S}$ is convex.

To prove the convexity of the interior $\mathring{S}$, we must show that for any two points $x, y \in \mathring{S}$ and for any $\lambda \in [0,1]$ the point $z := \varphi_\lambda(x,y) \in \mathring{S}$.

By definition of interior points of S, there exists a neighborhood U of the origin in X such that $x + U \subseteq S$ and $y + U \subseteq S$. Then we claim that $z + U \subseteq S$. This is indeed so, since for any element $u \in U$ we can write $z + u$ in the following form:

$$z + u = \lambda x + (1-\lambda)y + \lambda u + (1-\lambda)u = \lambda(x+u) + (1-\lambda)(y+u)$$

and since both vectors $x + u$ and $y + u$ belong to S, so does $z + u$. Hence, $z + U \subseteq S$ and so $z \in \mathring{S}$, which proves the convexity of $\mathring{S}$. □

Definition 4.1.8. *A subset T of a t.v.s. is called a* barrel *or* barrelled *if T has the following properties:*

1. *T is absorbing*
2. *T is absolutely convex*
3. *T is closed*

Proposition 4.1.9. *Every neighborhood of the origin in a t.v.s. is contained in a neighborhood of the origin which is a barrel.*

Proof.
Let U be a neighbourhood of the origin and define

$$T(U) := \overline{conv\left(B(U)\right)}, \text{ where } B(U) := \bigcup_{\lambda\in\mathbb{K},|\lambda|\leq 1} \lambda U.$$

Clearly, $U \subseteq T(U)$. Therefore, $T(U)$ is a neighbourhood of the origin and so it is absorbing by Theorem 2.1.10-4). By construction, $T(U)$ is also closed and convex as closure of a convex set (see Proposition 4.1.7). To prove that $T(U)$ is a barrel it remains to show that it is balanced.

Now $B(U)$ is balanced, because for any $x \in B(U)$ we have $x \in \lambda U$ for some $\lambda \in \mathbb{K}$ with $|\lambda| \leq 1$ and so $\mu x \in \mu\lambda U \in B(U)$ for all $\mu \in \mathbb{K}$ with $|\mu| \leq 1$. Then, by Proposition 4.1.5-h) and Proposition 2.1.13-a), $T(U)$ is also balanced.

□

Corollary 4.1.10. *Every neighborhood of the origin in a t.v.s. is contained in a neighborhood of the origin which is absolutely convex.*

Note that the converse of Proposition 4.1.9 does not hold in any t.v.s.. Indeed, not every neighborhood of the origin contains another one which is a barrel. This means that not every t.v.s. has a basis of neighbourhoods consisting of barrels. However, this is true for any locally convex t.v.s.

Definition 4.1.11. *A t.v.s. X is said to be* locally convex *(l.c.) if there is a basis of neighborhoods of the origin in X consisting of convex sets.*

Locally convex spaces are by far the most important class of t.v.s. and we will present later on several examples of such t.v.s.. For the moment let us focus on the properties of the filter of neighbourhoods of locally convex spaces.

Proposition 4.1.12. *A locally convex t.v.s. always has a basis of neighbourhoods of the origin consisting of open absorbing absolutely convex subsets.*

Proof.
Let X be a locally convex t.v.s. and N a neighbourhood of the origin in X. Since X is locally convex, there exists W convex neighbourhood of the origin in X s.t. $W \subseteq N$. Moreover, by Theorem 2.1.10-5, there exists U balanced neighbourhood of the origin in X s.t. $U \subseteq W$. Let us keep the notation of the previous proposition $B(U) := \bigcup_{\lambda\in\mathbb{K},|\lambda|\leq 1} \lambda U$. The balancedness of U implies that $U = B(U)$. Then, using that W is a convex set containing U, we get

$$O := conv(B(U)) = conv(U) \subseteq W \subseteq N$$

and so $\mathring{O} \subseteq N$. Hence, the conclusion holds because $\mathring{O}$ is clearly an open convex neigbourhood of the origin in X and it is also balanced by Proposition 2.1.13-b) since $o \in \mathring{O}$ and O is balanced (by Proposition 4.1.5-h)). □

Similarly, we get that

Proposition 4.1.13. *A locally convex t.v.s. always has a basis of neighbourhoods of the origin consisting of barrels.*

Proof.
Let X be a locally convex t.v.s. and N a neighbourhood of the origin in X. We know that every t.v.s. has a basis of closed neighbourhoods of the origin (see Corollary 2.1.14-a)). Then there exists V closed neighbourhood of the origin in X s.t. $V \subseteq N$. Since X is locally convex, then there exists W convex neighbourhood of the origin in X s.t. $W \subseteq V$. Moreover, by Theorem 2.1.10-5), there exists U balanced neighbourhood of the origin in X s.t. $U \subseteq W$. Summing up we have: $U \subseteq W \subseteq V \subseteq N$ for some U, W, V neighbourhoods of the origin s.t. U balanced, W convex and V closed. Let us keep the notation of the previous proposition $B(U) := \bigcup_{\lambda\in\mathbb{K},|\lambda|\leq 1} \lambda U$. The balancedness of U implies that $U = B(U)$. Then, using that W is a convex set containing U, we get

$$conv(B(U)) = conv(U) \subseteq W$$

Passing to the closures and using that V is closed, we get

$$T(U) = \overline{conv(U)} \subseteq \overline{W} \subseteq \overline{V} = V \subseteq N.$$

Hence, the conclusion holds because we have already showed in Proposition 4.1.9 that $T(U)$ is a barrelled neighbourhood of the origin in X. □

We can then characterize the class of locally convex t.v.s in terms of absorbing absolutely convex neighbourhoods of the origin.

Theorem 4.1.14. *If X is a l.c. t.v.s. then there exists a basis $\mathcal{B}$ of neighbourhoods of the origin consisting of absorbing absolutely convex subsets s.t.*
a) $\forall U, V \in \mathcal{B}$, $\exists W \in \mathcal{B}$ s.t. $W \subseteq U \cap V$
b) $\forall U \in \mathcal{B}$, $\forall\, \rho > 0$, $\exists W \in \mathcal{B}$ s.t. $W \subseteq \rho U$
Conversely, if $\mathcal{B}$ is a collection of absorbing absolutely convex subsets of a vector space X s.t. a) and b) hold, then there exists a unique topology compatible with the linear structure of X s.t. $\mathcal{B}$ is a basis of neighbourhoods of the origin in X for this topology (which is necessarily locally convex).

Proof. (Christmas Assignment) □

In particular, the collection $\mathcal{M}$ of all multiples ρU of an absorbing absolutely convex subset U of a vector space X is a basis of neighborhoods of the origin for a locally convex topology on X compatible with the linear structure (this ceases to be true, in general, if we relax the conditions on U).

Proof. First of all, let us observe that for any $\rho \in \mathbb{K} \setminus \{0\}$, we have that ρU is absorbing and absolutely convex since U has such properties.

For any $A, B \in \mathcal{M}$, there exist $\lambda, \mu \in \mathbb{K} \setminus \{0\}$ s.t. $A = \lambda U$ and $B = \mu U$. W.l.o.g. we can assume $|\lambda| \leq |\mu|$ and so $\frac{\lambda}{\mu} U \subseteq U$, i.e. $A \subseteq B$. Hence, a) and b) in Theorem 4.1.14 are fulfilled since $A \cap B = A \in \mathcal{M}$ and, for any $\rho \in \mathbb{K}$, $\rho A = \rho \lambda U \in \mathcal{M}$.

Therefore, Theorem 4.1.14 ensures that $\mathcal{M}$ is a basis of neighbourhoods of the origin of a topology which makes X into a l.c. t.v.s..

□

4.2 Connection to seminorms

In applications it is often useful to define a locally convex space by means of a system of seminorms. In this section we will investigate the relation between locally convex t.v.s. and seminorms.

Definition 4.2.1. *Let X be a vector space. A function $p : X \to \mathbb{R}$ is called a* seminorm *if it satisfies the following conditions:*

1. *p is subadditive: $\forall x, y \in X$, $p(x + y) \leq p(x) + p(y)$.*
2. *p is positively homogeneous: $\forall x, y \in X$, $\forall \lambda \in \mathbb{K}, p(\lambda x) = |\lambda| p(x)$.*

Definition 4.2.2.
A seminorm p on a vector space X is a norm *if $p^{-1}(\{0\}) = \{o\}$ (i.e. if $p(x) = 0$ implies $x = o$).*

Proposition 4.2.3. *Let p be a seminorm on a vector space X. Then the following properties hold:*

- *p is symmetric.*
- *$p(o) = 0$.*
- *$|p(x) - p(y)| \leq p(x - y)$, $\forall x, y \in X$.*
- *$p(x) \geq 0$, $\forall x \in X$.*
- *$Ker(p)$ is a linear subspace of X.*

Proof.

- The symmetry of p directly follows from the positive homogeneity of p. Indeed, for any $x \in X$ we have

$$p(-x) = p(-1 \cdot x) = |-1| p(x) = p(x).$$

- Using again the positive homogeneity of p we get that $p(o) = p(0 \cdot x) = 0 \cdot p(x) = 0$.
- For any $x, y \in X$, the subadditivity of p guarantees the following inequalities:

$$p(x) = p(x-y+y) \leq p(x-y)+p(y) \quad \text{and} \quad p(y) = p(y-x+x) \leq p(y-x)+p(x)$$

which establish the third property.
- The previous property directly gives the nonnegativity of p. In fact, for any $x \in X$ we get

$$0 \leq |p(x) - p(o)| \leq p(x - o) = p(x).$$

- Let $x, y \in Ker(p)$ and $\alpha, \beta \in \mathbb{K}$. Then

$$p(\alpha x + \beta y) \leq |\alpha| p(x) + |\beta| p(y) = 0$$

which implies,by the nonnegativity of p, that $p(\alpha x + \beta y) = 0$. Hence, we have $\alpha x + \beta y \in Ker(p)$.

□

Examples 4.2.4.

a) *Suppose $X = \mathbb{R}^n$ and let M be a linear subspace of X. Set for any $x \in X$*

$$q_M(x) := \inf_{m \in M} \|x - m\|$$

where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^n$, i.e. $q_M(x)$ is the distance from the point x to M in the usual sense. If $dim(M) \geq 1$ then q_M is a seminorm and not a norm (M is exactly the kernel of q_M). When $M = \{o\}$, $q_M(\cdot) = \|\cdot\|$.

b) *Let $\mathcal{C}(\mathbb{R})$ be the vector space of all real valued continuous functions on the real line. For any bounded interval $[a, b]$ with $a, b \in \mathbb{R}$ and $a < b$, we define for any $f \in \mathcal{C}(\mathbb{R})$:*

$$q_{[a,b]}(f) := \sup_{a \leq t \leq b} |f(t)|.$$

$q_{[a,b]}$ is a seminorm but is never a norm because it might be that $f(t) = 0$ for all $t \in [a, b]$ (and so that $q_{[a,b]}(f) = 0$) but $f \not\equiv 0$. Other seminorms are the following ones:

$$q(f) := |f(0)| \quad \text{and} \quad q_p(f) := \left(\int_a^b |f(t)|^p dt \right)^{\frac{1}{p}} \quad \text{for } 1 \leq p < \infty.$$

Note that if $0 < p < 1$ then q_p is not subadditive and so it is not a seminorm (see Christmas assignment).

c) Let X be a vector space on which is defined a nonnegative sesquilinear Hermitian form $B : X \times X \to \mathbb{K}$. Then the function

$$p_B(x) := B(x, x)^{\frac{1}{2}}$$

is a seminorm. q_B is a norm if and only if B is positive definite (i.e. $B(x, x) > 0$, $\forall x \neq o$).