c) *Let X be a vector space on which is defined a nonnegative sesquilinear Hermitian form $B : X \times X \to \mathbb{K}$. Then the function*

$$p_B(x) := B(x,x)^{\frac{1}{2}}$$

is a seminorm. q_B is a norm if and only if B is positive definite (i.e. $B(x,x) > 0$, $\forall x \neq o$).

Seminorms on vector spaces are strongly related to a special kind of functionals, i.e. *Minkowski functionals*. Let us investigate more in details such a relation. Note that we are still in the realm of vector spaces with no topology!

Definition 4.2.5. *Let X be a vector space and A a non-empty subset of X. We define the* Minkowski functional (or gauge) *of A to be the mapping:*

$$\begin{array}{rccl} p_A : & X & \to & \mathbb{R} \\ & x & \mapsto & p_A(x) := \inf\{\lambda > 0 : x \in \lambda A\} \end{array}$$

(where $p_A(x) = \infty$ if the set $\{\lambda > 0 : x \in \lambda A\}$ is empty).

It is then natural to ask whether there exists a class of subsets for which the associated Minkowski functionals are actually seminorms. The answer is positive for a class of subsets which we have already encountered in the previous section, namely for absorbing absolutely convex subsets. Actually we have even more as established in the following lemma.

Notation 4.2.6. *Let X be a vector space and p a seminorm on X. The sets*

$$\mathring{U}_p = \{x \in X : p(x) < 1\} \text{ and } U_p = \{x \in X : p(x) \leq 1\}.$$

are said to be, respectively, the closed and the open unit semiball *of p.*

Lemma 4.2.7. *Let X be a vector space. If A is a non-empty subset of X which is absorbing and absolutely convex, then the associated Minkowski functional p_A is a seminorm and $\mathring{U}_{p_A} \subseteq A \subseteq U_{p_A}$. Conversely, if q is a seminorm on X then $\mathring{U}_q$ is an absorbing absolutely convex set and $q = p_{\mathring{U}_q}$.*

Proof. Let A be a non-empty subset of X which is absorbing and absolutely convex and denote by p_A the associated Minkowski functional. We want to show that p_A is a seminorm.

- First of all, note that $p_A(x) < \infty$ for all $x \in X$ because A is absorbing. Indeed, by definition of absorbing set, for any $x \in X$ there exists $\rho_x > 0$ s.t. for all $\lambda \in \mathbb{K}$ with $|\lambda| \leq \rho_x$ we have $\lambda x \in A$ and so the set $\{\lambda > 0 : x \in \lambda A\}$ is never empty i.e. p_A has only finite nonnegative values. Moreover, since $o \in A$, we also have that $o \in \lambda A$ for any $\lambda \in \mathbb{K}$ and so $p_A(o) = \inf\{\lambda > 0 : o \in \lambda A\} = 0$.

- The balancedness of A implies that p_A is positively homogeneous. Since we have already showed that $p_A(o) = 0$ it remains to prove the positive homogeneity of p_A for non-null scalars. Since A is balanced we have that for any $x \in X$ and for any $\xi, \lambda \in \mathbb{K}$ with $\xi \neq 0$ the following holds:
$$\xi x \in \lambda A \text{ if and only if } x \in \frac{\lambda}{|\xi|}A. \tag{4.1}$$
Indeed, A balanced guarantees that $\xi A = |\xi| A$ and so $x \in \frac{\lambda}{|\xi|}A$ is equivalent to $\xi x \in \lambda \frac{\xi}{|\xi|}A = \lambda A$. Using (4.1), we get that for any $x \in X$ and for any $\xi \in \mathbb{K}$ with $\xi \neq 0$:
$$\begin{aligned} p_A(\xi x) &= \inf\{\lambda > 0 : \xi x \in \lambda A\} \\ &= \inf\left\{\lambda > 0 : x \in \frac{\lambda}{|\xi|}A\right\} \\ &= \inf\left\{|\xi|\frac{\lambda}{|\xi|} > 0 : x \in \frac{\lambda}{|\xi|}A\right\} \\ &= |\xi| \inf\{\mu > 0 : x \in \mu A\} = |\xi| p_A(x). \end{aligned}$$
- The convexity of A ensures the subadditivity of p_A. Take $x, y \in X$. By definition of Minkowski functional, for every $\varepsilon > 0$ there exists $\lambda, \mu > 0$ s.t.
$$\lambda \leq p_A(x) + \varepsilon \text{ and } x \in \lambda A$$
and
$$\mu \leq p_A(y) + \varepsilon \text{ and } y \in \mu A.$$
Then, by the convexity of A, we obtain that $\frac{\lambda}{\lambda+\mu}A + \frac{\mu}{\lambda+\mu}A \subseteq A$, i.e. $\lambda A + \mu A \subseteq (\lambda + \mu)A$, and therefore $x + y \in (\lambda + \mu)A$. Hence:
$$p_A(x + y) = \inf\{\delta > 0 : x + y \in \delta A\} \leq \lambda + \mu \leq p_A(x) + p_A(y) + 2\varepsilon$$
which proves the subadditivity of p_A since ε is arbitrary.

We can then conclude that p_A is a seminorm. Furthermore, we have the following inclusions:
$$\mathring{U}_{p_A} \subseteq A \subseteq U_{p_A}.$$
In fact, if $x \in \mathring{U}_{p_A}$ then $p_A(x) < 1$ and so there exists $0 \leq \lambda < 1$ s.t. $x \in \lambda A$. Since A is balanced, for such λ we have $\lambda A \subseteq A$ and therefore $x \in A$. On the other hand, if $x \in A$ then clearly $1 \in \{\lambda > 0 : x \in \lambda A\}$ which gives $p_A(x) \leq 1$ and so $x \in U_{p_A}$.

Conversely, let us take any seminorm q on X. Let us first show that $\mathring{U}_q$ is absorbing and absolutely convex and then that q coincides with the Minkowski functional associated to $\mathring{U}_q$.

- $\mathring{U}_q$ is absorbing.
  Let x be any point in X. If $q(x) = 0$ then clearly $x \in \mathring{U}_q$. If $q(x) > 0$, we can take $0 < \rho < \frac{1}{q(x)}$ and then for any $\lambda \in \mathbb{K}$ s.t. $|\lambda| \leq \rho$ the positive homogeneity of q implies that $q(\lambda x) = |\lambda| q(x) \leq \rho q(x) < 1$, i.e. $\lambda x \in \mathring{U}_q$.
- $\mathring{U}_q$ is balanced.
  For any $x \in \mathring{U}_q$ and for any $\lambda \in \mathbb{K}$ with $|\lambda| \leq 1$, again by the positive homogeneity of q, we get: $q(\lambda x) = |\lambda| q(x) \leq q(x) < 1$ i.e. $\lambda x \in \mathring{U}_q$.
- $\mathring{U}_q$ is convex.
  For any $x, y \in \mathring{U}_q$ and any $t \in [0, 1]$, by both the properties of seminorm, we have that $q(tx + (1-t)y) \leq tq(x) + (1-t)q(y) < t + 1 - t = 1$ i.e. $tx + (1-t)y \in \mathring{U}_q$.

Moreover, for any $x \in X$ we easily see that

$$p_{\mathring{U}_q}(x) = \inf\{\lambda > 0 : x \in \lambda \mathring{U}_q\} = \inf\{\lambda > 0 : q(x) < \lambda\} = q(x).$$

□

We are now ready to see the connection between seminorms and locally convex t.v.s..

Definition 4.2.8. *Let X be a vector space and $\mathcal{P} := \{p_i\}_{i \in I}$ a family of seminorms on X. The coarsest topology $\tau_{\mathcal{P}}$ on X s.t. each p_i is continuous is said to be the* topology induced or generated by the family of seminorms $\mathcal{P}$.

Theorem 4.2.9. *Let X be a vector space and $\mathcal{P} := \{p_i\}_{i \in I}$ a family of seminorms. Then the topology induced by the family $\mathcal{P}$ is the unique topology making X into a locally convex t.v.s. and having as a basis of neighbourhoods of the origin in X the following collection:*

$$\mathcal{B} := \Big\{ \{x \in X : p_{i_1}(x) < \varepsilon, \ldots, p_{i_n}(x) < \varepsilon\} : i_1, \ldots, i_n \in I, n \in \mathbb{N}, \varepsilon > 0, \epsilon \in \mathbb{R} \Big\}.$$

Viceversa, the topology of an arbitrary locally convex t.v.s. is always induced by a family of seminorms (often called generating*).*

Proof. Let us first show that the collection $\mathcal{B}$ is a basis of neighbourhoods of the origin for the unique topology τ making X into a locally convex t.v.s. by using Theorem 4.1.14 and then let us prove that τ actually coincides with the topology induced by the family $\mathcal{P}$.

For any $i \in I$ and any $\varepsilon > 0$, consider the set $\{x \in X : p_i(x) < \varepsilon\} = \varepsilon \mathring{U}_{p_i}$. This is absorbing and absolutely convex, since we have already showed above that $\mathring{U}_{p_i}$ fulfills such properties. Therefore, any element of $\mathcal{B}$ is an absorbing absolutely convex subset of X as finite intersection of absorbing absolutely convex sets. Moreover, both properties a) and b) of Theorem 4.1.14 are clearly satisfied by $\mathcal{B}$. Hence, Theorem 4.1.14 guarantees that there exists a unique topology τ on X s.t. (X, τ) is a locally convex t.v.s. and $\mathcal{B}$ is a basis of neighbourhoods of the origin for τ.

Let us consider (X, τ). Then for any $i \in I$, the seminorm p_i is continuous, because for any $\varepsilon > 0$ we have $p_i^{-1}([0, \varepsilon[) = \{x \in X : p_i(x) < \varepsilon\} \in \mathcal{B}$ which means that $p_i^{-1}([0, \varepsilon[)$ is a neighbourhood of the origin in (X, τ). Therefore, the topology $\tau_{\mathcal{P}}$ induced by the family $\mathcal{P}$ is by definition coarser than τ. On the other hand, each p_i is also continuous w.r.t. $\tau_{\mathcal{P}}$ and so $\mathcal{B} \subseteq \tau_{\mathcal{P}}$. But $\mathcal{B}$ is a basis for τ, then necessarily τ is coarser than $\tau_{\mathcal{P}}$. Hence, $\tau \equiv \tau_{\mathcal{P}}$.

Viceversa, let us assume that (X, τ) is a locally convex t.v.s.. Then by Theorem 4.1.14 there exists a basis $\mathcal{N}$ of neighbourhoods of the origin in X consisting of absorbing absolutely convex sets s.t. the properties a) and b) in Theorem 4.1.14 are fulfilled. W.l.o.g. we can assume that they are open. Consider now the family $\mathcal{S} := \{p_N : N \in \mathcal{N}\}$. By Lemma 4.2.7, we know that each p_N is a seminorm and that $\mathring{U}_{p_N} \subseteq N$. Let us show that for any $N \in \mathcal{N}$ we have actually that $N = \mathring{U}_{p_N}$. Since any $N \in \mathcal{N}$ is open and the scalar multiplication is continuous we have that for any $x \in N$ there exists $0 < t < 1$ s.t. $x \in tN$ and so $p_N(x) \leq t < 1$, i.e. $x \in \mathring{U}_{p_N}$.

We want to show that the topology $\tau_{\mathcal{S}}$ induced by the family $\mathcal{S}$ coincides with original topology τ on X. We know from the first part of the proof how to construct a basis for a topology induced by a family of seminorms. In fact, a basis of neighbourhoods of the origin for $\tau_{\mathcal{S}}$ is given by

$$\mathcal{B} := \left\{ \bigcap_{i=1}^{n} \{x \in X : p_{N_i}(x) < \varepsilon\} : N_1, \ldots, N_n \in \mathcal{N}, n \in \mathbb{N}, \varepsilon > 0, \epsilon \in \mathbb{R} \right\}.$$

For any $N \in \mathcal{N}$ we showed that $N = \mathring{U}_{p_N} \in \mathcal{B}$ so by Hausdorff criterion we get $\tau \subseteq \tau_{\mathcal{S}}$. Also for any $B \in \mathcal{B}$ we have $B = \cap_{i=1}^{n} \varepsilon \mathring{U}_{p_{N_i}} = \cap_{i=1}^{n} \varepsilon N_i$ for some $n \in \mathbb{N}$, $N_1, \ldots, N_n \in \mathcal{N}$ and $\varepsilon > 0$. Then property b) of Theorem 4.1.14 for $\mathcal{N}$ implies that for each $i = 1, \ldots, n$ there exists $V_i \in \mathcal{N}$ s.t. $V_i \subseteq \varepsilon N_i$ and so by the property a) of $\mathcal{N}$ we have that there exists $V \in \mathbb{N}$ s.t. $V \subseteq \cap_{i=1}^{n} V_i \subseteq B$. Hence, by Hausdorff criterion $\tau_{\mathcal{S}} \subseteq \tau$. □

This result justifies why several authors define a locally convex space to be a t.v.s whose topology is induced by a family of seminorms (which is now evidently equivalent to Definition 4.1.11).

In the previous proofs we have used some interesting properties of semiballs in a vector space. For convenience, we collect them here together with some further ones which we will repeatedly use in the following.

Proposition 4.2.10. *Let X be a vector space and p a seminorm on X. Then:*

a) $\mathring{U}_p$ *is absorbing and absolutely convex.*

b) $\forall\, r > 0$, $r\mathring{U}_p = \{x \in X : p(x) < r\} = \mathring{U}_{\frac{1}{r}p}$.

c) $\forall\, x \in X$, $x + \mathring{U}_p = \{y \in X : p(y - x) < 1\}$.

d) If q is also a seminorm on X then: $p \leq q$ if and only if $\mathring{U}_q \subseteq \mathring{U}_p$.

e) If $n \in \mathbb{N}$ and $s_1, \dots, s_n$ are seminorms on X, then their maximum *s defined as $s(x) := \max_{i=1,\dots,n} s_i(x)$, $\forall x \in X$ is also seminorm on X and $\mathring{U}_s = \bigcap_{i=1}^n \mathring{U}_{s_i}$.*

All the previous properties also hold for closed semballs.

Proof.

a) This was already proved as part of Lemma 4.2.7.

b) For any $r > 0$, we have
$$r\mathring{U}_p = \{rx \in X : p(x) < 1\} = \underbrace{\{y \in X : \frac{1}{r}p(y) < 1\}}_{\mathring{U}_{\frac{1}{r}p}} = \{y \in X : p(y) < r\}.$$

c) For any $x \in X$, we have
$$x + \mathring{U}_p = \{x + z \in X : p(z) < 1\} = \{y \in X : p(y - x) < 1\}.$$

d) Suppose that $p \leq q$ and take any $x \in \mathring{U}_q$. Then we have $q(x) < 1$ and so $p(x) \leq q(x) < 1$, i.e. $x \in \mathring{U}_p$. Viceversa, suppose that $\mathring{U}_q \subseteq \mathring{U}_p$ holds and take any $x \in X$. We have that either $q(x) > 0$ or $q(x) = 0$. In the first case, for any $0 < \varepsilon < 1$ we get that $q\left(\frac{\varepsilon x}{q(x)}\right) = \varepsilon < 1$. Then $\frac{\varepsilon x}{q(x)} \in \mathring{U}_q$ which implies by our assumption that $\frac{\varepsilon x}{q(x)} \in \mathring{U}_p$ i.e. $p\left(\frac{\varepsilon x}{q(x)}\right) < 1$. Hence, $\varepsilon p(x) < q(x)$ and so when $\varepsilon \to 1$ we get $p(x) \leq q(x)$. If instead we are in the second case that is when $q(x) = 0$, then we claim that also $p(x) = 0$. Indeed, if $p(x) > 0$ then $q\left(\frac{x}{p(x)}\right) = 0$ and so $\frac{x}{p(x)} \in \mathring{U}_q$ which implies by our assumption that $\frac{x}{p(x)} \in \mathring{U}_p$, i.e. $p(x) < p(x)$ which is a contradiction.

e) It is easy to check, using basic properties of the maximum, that the subadditivity and the positive homogeneity of each s_i imply the same properties for s. In fact, for any $x, y \in X$ and for any $\lambda \in \mathbb{K}$ we get:

- $$\begin{aligned} s(x+y) &= \max_{i=1,\dots,n} s_i(x+y) \leq \max_{i=1,\dots,n} (s_i(x) + s_i(y)) \\ &\leq \max_{i=1,\dots,n} s_i(x) + \max_{i=1,\dots,n} s_i(y) = s(x) + s(y) \end{aligned}$$
- $s(\lambda x) = \max_{i=1,\dots,n} s_i(\lambda x) = |\lambda| \max_{i=1,\dots,n} s_i(x) = |\lambda| s(x)$.

Moreover, if $x \in \mathring{U}_s$ then $\max_{i=1,\dots,n} s_i(x) < 1$ and so for all $i = 1, \dots, n$ we have $s_i(x) < 1$, i.e. $x \in \bigcap_{i=1}^n \mathring{U}_{s_i}$. Conversely, if $x \in \bigcap_{i=1}^n \mathring{U}_{s_i}$ then for all $i = 1, \dots, n$ we have $s_i(x) < 1$. Since $s(x)$ is the maximum over a finite number of terms, it will be equal to $s_j(x)$ for some $j \in \{1, \dots, n\}$ and therefore $s(x) = s_j(x) < 1$, i.e. $x \in \mathring{U}_s$. □