Universität Konstanz
Fachbereich Mathematik und Statistik
Dr. Maria Infusino
Patrick Michalski

TOPOLOGICAL VECTOR SPACES–WS 2018/19

Exercise Sheet 1

This assignment is due by November 13th by 15:00 and will be discussed in the following tutorials: November 16th (Group A) and November 19th (Group B). Please, hand in your solutions in the Postbox 11 (near F411). If you should have any problem in solving the exercises, please take advantage of the Fragestunde *on Wednesday 13:30–14:30 in room F408.*

1) **a)** Suppose X is a topological space, and for every $p \in X$ there exists a continuous functional $f : X \to \mathbb{R}$ such that $f^{-1}(\{0\}) = \{p\}$. Then X is Hausdorff.

b) Let X be a set endowed with the trivial topology and Y be any topological space. If Y is Hausdorff, then the only continuous maps $h : X \to Y$ are constant maps.

2) Show the following statements using just the definition of t.v.s..

a) Every normed vector space $(X, \|\cdot\|)$ endowed with the topology given by the metric induced by the norm is a t.v.s.. (Hint: use the collection $\{B_r(x_0) := \{x \in X : \|x - x_0\| < r\} : r \in \mathbb{R}^+, x_0 \in X\}$ as a base of the topology).

b) Consider the real vector space $\mathbb{R}$ endowed with the lower limit topology τ generated by the base $B = \{[a, b) : a < b\}$. Show that $(\mathbb{R}, \tau)$ is not a t.v.s..

c) Let us consider on $\mathbb{R}$ the metric:

$$d_h(x, y) := |h(x) - h(y)|, \forall x, y \in \mathbb{R},$$

where h is the following function on $\mathbb{R}$:

$$h(x) := \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x = 1 \\ x & \text{otherwise} \end{cases} .$$

Then the metric real vector space $(\mathbb{R}, d_h)$ is not a t.v.s. (here the field of scalars is also $\mathbb{R}$ but endowed with the usual topology given by the modulus $|\cdot|$).

3) Prove the following statements.

a) The filter $\mathcal{F}(x)$ of neighbourhoods of the point x in a t.v.s. X coincides with the family of the sets $O + x$ for all $O \in \mathcal{F}(o)$, where $\mathcal{F}(o)$ is the filter of neighbourhoods of the origin o (i.e. the neutral element of the vector addition).

b) If B is a balanced subset of a t.v.s. X then so is $\bar{B}$.

c) If B is a balanced subset of a t.v.s. X and $o \in \mathring{B}$ then $\mathring{B}$ is balanced.

4) Prove the following statements.

a) Every t.v.s. has always a base of closed neighbourhoods of the origin.

b) Every t.v.s. has always a base of balanced absorbing neighbourhoods of the origin. In particular, it has always a base of closed balanced absorbing neighbourhoods of the origin.

c) Proper linear subspaces of a t.v.s. are never absorbing. In particular, if M is an open linear subspace of a t.v.s. X, then $M = X$.