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Fachbereich Mathematik und Statistik
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TOPOLOGICAL VECTOR SPACES–WS 2018/19

Exercise Sheet 3

This assignment is due by December 11th by 15:00 and will be discussed in the following tutorials: December 14th (Group A) and December 17th (Group B). Please, hand in your solutions in the Postbox 11 (near F411). If you should have any problem in solving the exercises, please take advantage of the Fragestunde *on Wednesday 13:30–14:30 in room F408.*

1) Let X be a t.v.s.. Assume that there exists a countable basis $\mathcal{B}$ of neighbourhoods of the origin in X. Prove the following statements:

a) X is complete if and only if X is sequentially complete.

b) Suppose additionally that Y is another t.v.s. (not necessarily with a countable basis).
A mapping $f : X \to Y$ (not necessarily linear) is continuous if and only if it is sequentially continuous.

Recall that a mapping f from a topological space X into a topological space Y is said to be *sequentially continuous* if for every sequence $\{x_n\}_{n\in\mathbb{N}}$ convergent to a point $x \in X$ the sequence $\{f(x_n)\}_{n\in\mathbb{N}}$ converges to $f(x)$ in Y.

2) Let $\mathcal{C}(\mathbb{R})$ be the vector space of real valued functions defined and continuous on the real line. For any $\varepsilon \in \mathbb{R}^+$ and any $n \in \mathbb{N}$, set $N_{\varepsilon,n} := \left\{ f \in \mathcal{C}(\mathbb{R}) : \sup_{|t|\leq n} |f(t)| \leq \varepsilon \right\}$ and prove that:

a) The collection $\{N_{\varepsilon,n} : \varepsilon \in \mathbb{R}^+, n \in \mathbb{N}\}$ is a basis of neighbourhoods of the origin for a topology τ on $\mathcal{C}(\mathbb{R})$ which is compatible with its linear structure.

b) The t.v.s. $(\mathcal{C}(\mathbb{R}), \tau)$ is a complete Hausdorff space [Hint: use Exercise 1].

3) Let X be a t.v.s. over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ with the usual topology given by the modulus, and L a linear functional on X. Assume $L(x) \neq 0$ for some $x \in X$. Show that the following are equivalent:

a) L is continuous.

b) The null space $\ker(L)$ is closed in X.

c) $\ker(L)$ is not dense in X.

d) L is bounded in some neighbourhood of the origin in X, i.e. $\exists\, V \in \mathcal{F}(o)$ s.t. $\sup_{x\in V} |L(x)| < \infty$.

4) Keeping in mind the following definition:

Definition 1. *Let X and Y be two t.v.s. and let A be a subset of X. A mapping $f : A \to Y$ is said to be* uniformly continuous *if for every neighbourhood V of the origin in Y, there exists a neighbourhood U of the origin in X such that for all pairs of elements $x_1, x_2 \in A$ the implication $x_1 - x_2 \in U \Rightarrow f(x_1) - f(x_2) \in V$ holds.*

Show that if X and Y are two t.v.s. and A is a subset of X then the following hold:

a) Any uniformly continuous map $f : A \to Y$ is continuous at every point of A.

b) If $f : A \to Y$ is uniformly continuous, then the image under f of a Cauchy filter on A is a Cauchy filter on Y.

c) If A is a linear subspace of X, then every continuous linear map from A to Y is uniformly continuous.