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Fachbereich Mathematik und Statistik
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TOPOLOGICAL VECTOR SPACES–WS 2018/19

Christmas Assignment

This assignment is due by January 8th by 15:00 and will be discussed in the following tutorials: January 11th (Group A) and January 14th (Group B). Please, hand in your solutions in the Postbox 11 (near F411). If you should have any problem in solving the exercises, please take advantage of the Fragestunde *on Wednesday 13:30–14:30 in room F408.*

1) Let X be a Hausdorff t.v.s. over $\mathbb{R}$ and X^* its algebraic dual. Provide X^* with the topology of pointwise convergence in X. A basis of neighborhoods of the origin in this topology is provided by the sets

$$W(S,\varepsilon) := \{\ell \in X^* : \sup_{x\in S} |\ell(x)| \leq \varepsilon\},$$

where S ranges over the family of finite subsets of X and $\varepsilon \in \mathbb{R}^+$. Prove that X^* is complete.

2) Prove the following characterization of locally convex t.v.s (i.e. Theorem 4.1.14 in the lecture notes)

Theorem 1. *If X is a l.c. t.v.s., then there exists a basis $\mathcal{B}$ of neighbourhoods of the origin consisting of absorbing absolutely convex subsets s.t.*

a) $\forall U, V \in \mathcal{B}, \exists W \in \mathcal{B}$ *s.t.* $W \subseteq U \cap V$

b) $\forall U \in \mathcal{B}, \forall \rho > 0, \exists W \in \mathcal{B}$ *s.t.* $W \subseteq \rho U$

Conversely, if $\mathcal{B}$ is a collection of absorbing absolutely convex subsets of a vector space X s.t. a) and b) hold, then there exists a unique topology compatible with the linear structure of X s.t. $\mathcal{B}$ is a basis of neighbourhoods of the origin in X for this topology (which is necessarily locally convex).

3) Let $\mathcal{C}(\mathbb{R})$ be the vector space of all real valued continuous functions on the real line. For any bounded interval $[a,b]$ with $a < b$ and any $p > 0$, we define:

$$q_p(f) := \left(\int_a^b |f(t)|^p dt\right)^{\frac{1}{p}}, \ \forall\, f \in \mathcal{C}(\mathbb{R}).$$

Show that for any $1 \leq p < \infty$ the function q_p is a seminorm but that if $0 < p < 1$ then q_p is not a seminorm.

4) Let $0 < p < 1$ and consider the vector space

$$\ell_p := \{(x_i)_{i\in\mathbb{N}} : \forall i \in \mathbb{N}, x_i \in \mathbb{R} \text{ and } \sum_{i=1}^{\infty} |x_i|^p < \infty\}.$$

For any $x, y \in \ell_p$ define $d(x,y) := |x-y|_p$, where for any $z := (x_i)_{i\in\mathbb{N}} \in \ell_p$ we set $|z|_p := \sum_{i=1}^{\infty} |z_i|^p$. Show that the t.v.s. given by ℓ_p endowed with the topology induced by d is not locally convex.