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Fachbereich Mathematik und Statistik
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TOPOLOGICAL VECTOR SPACES–WS 2018/19

Exercise Sheet 5

This assignment is due by January 22nd by 15:00 and will be discussed in the following tutorials: January 25th (Group A) and January 28th (Group B). Please, hand in your solutions in the Postbox 11 (near F411). If you should have any problem in solving the exercises, please take advantage of the Fragestunde *on Wednesday 13:30–14:30 in room F408.*

1) Consider the following theorem (Theorem 4.2.12 in the lecture notes) about the comparison of locally convex topologies.

Theorem 1. *Let $\mathcal{P} = \{p_i\}_{i\in I}$ and $\mathcal{Q} = \{q_j\}_{j\in J}$ be two families of seminorms on the vector space X inducing respectively the topologies $\tau_{\mathcal{P}}$ and $\tau_{\mathcal{Q}}$, which both make X into a locally convex t.v.s.. Then $\tau_{\mathcal{P}}$ is finer than $\tau_{\mathcal{Q}}$ (i.e. $\tau_{\mathcal{Q}} \subseteq \tau_{\mathcal{P}}$) iff*

$$\forall q \in \mathcal{Q}\ \exists n \in \mathbb{N},\, i_1,\dots,i_n \in I,\, C>0 \text{ s.t. } Cq(x) \leq \max_{k=1,\dots,n} p_{i_k}(x),\ \forall x \in X. \tag{1}$$

a) Give an alternative proof of this result without using Proposition 4.2.11 in the lecture notes.

b) Show that the theorem still holds if we replace (1) with:

$$\forall q \in \mathcal{Q}\ \exists n \in \mathbb{N},\, i_1,\dots,i_n \in I,\, C>0 \text{ s.t. } Cq(x) \leq \sum_{k=1}^{n} p_{i_k}(x),\ \forall x \in X.$$

2) Fix some $d \in \mathbb{N}$ and let $\mathcal{C}^\infty(\mathbb{R}^d)$ be the space of infinitely differentiable functions endowed with the Hausdorff locally convex topology induced by the family $\mathcal{P} := \{p_{m,K} : m \in \mathbb{N}_0, K \subseteq \mathbb{R}^d \text{ compact}\}$, where $p_{m,K}(f) := \sup_{\beta\in\mathbb{N}_0^d, |\beta|\leq m} \sup_{x\in K} \left|(D^\beta f)(x)\right|$.

(a) Consider the Schwartz space or space of rapidly decreasing functions on $\mathbb{R}^d$ denoted by $\mathcal{S}(\mathbb{R}^d)$;

$$\mathcal{S}(\mathbb{R}^d) := \left\{ f \in \mathcal{C}^\infty(\mathbb{R}^d) : \sup_{x\in\mathbb{R}^d} \left|x^\alpha (D^\beta f)(x)\right| < \infty, \forall \alpha, \beta \in \mathbb{N}_0^d \right\}.$$

Show that the maps $q_{\alpha,\beta} : \mathcal{S}(\mathbb{R}^d) \to \mathbb{R}$ defined by $q_{\alpha,\beta}(f) := \sup_{x\in\mathbb{R}^d} \left|x^\alpha (D^\beta f)(x)\right|$ for $\alpha, \beta \in \mathbb{N}_0^d$ are seminorms and that they induce a topology τ_Q on $\mathcal{S}(\mathbb{R}^d)$ making it into a Hausdorff locally convex t.v.s..

(b) Endow $\mathcal{S}(\mathbb{R}^d) \subseteq \mathcal{C}^\infty(\mathbb{R}^d)$ with the subspace topology $\tau_{\mathcal{P}}^{\mathcal{S}}$ induced by $\tau_{\mathcal{P}}$ and show that the topology τ_Q is finer than $\tau_{\mathcal{P}}^{\mathcal{S}}$.

3) Let X be a locally convex t.v.s. whose topology τ is induced by a directed family of seminorms $\mathcal{P}$. Show that a basis of neighbourhoods of the origin in X for τ is given by $\mathcal{B} := \{r\mathring{U}_p : p \in \mathcal{P}, r > 0\}$, where $\mathring{U}_p := \{x \in X : p(x) < 1\}$.

4) Provide an alternative proof of the following result (Proposition 4.4.2 in the lecture notes) without using Theorem 4.1.14 but exploiting Proposition 4.4.1 and the results in Section 4.2.

Proposition 2. *The collection of all absorbing absolutely convex sets of a vector space X is a basis of neighbourhoods of the origin for the finest locally convex topology on X.*