Universität Konstanz
Fachbereich Mathematik und Statistik
Dr. Maria Infusino
Patrick Michalski

TOPOLOGICAL VECTOR SPACES–WS 2018/19

Exercise Sheet 6

This assignment is due by *February 5th by 15:00* *and the solutions will be discussed in the* ***unique tutorial for both Groups A and B on February 12th, 15:15–16.45, Room D301****. Please, hand in your solutions in the Postbox 11 (near F411). If you should have any problem in solving the exercises, please take advantage of the* Fragestunde *on Monday February 4th, 15:00–16:00 in room F408.*

1) Keeping in mind the definition of finite topology on a countable dimensional vector space (see Definition 4.5.1 in the lecture notes), prove the following statements.

a) Let X, Y be two infinite dimensional vector spaces of countable dimension each endowed with the corresponding finite topology. Then the finite topology on the product $X \times Y$ coincides with the product topology.

b) Let X be an infinite dimensional vector space with basis $\{x_n\}_{n\in\mathbb{N}}$ endowed with the finite topology τ_f and (Y, τ) any other topological space. For any $i \in \mathbb{N}$ set $X_i := \text{span}\{x_1, \dots, x_i\}$ so that $X = \bigcup_{i=1}^{\infty} X_i$. A map $f : X \to Y$ is continuous (w.r.t. τ_f and τ) if and only if for each $i \in \mathbb{N}$ the restriction $f|_{X_i}$ of f to X_i is continuous (w.r.t. the euclidean topology and τ).

c) Any countable dimensional vector space endowed with the finite topology is a t.v.s..
(*Hint*: use the properties (a) and (b))

2) Show that if X is a locally convex Hausdorff t.v.s with $X \neq \{o\}$, then for every $o \neq x_0 \in X$ there exists a continuous linear functional ℓ on X such that s.t. $\ell(x_0) \neq 0$.
(*Hint*: use Hahn-Banach Theorem)

3) Let $\mathbb{R}[x]$ denote the vector space of polynomials in the variable x and endow it with the finite topology τ_f. Consider the subset $T := \{p \in \mathbb{R}[x] : p(x) \geq 0 \text{ for all } x \in \mathbb{R}\}$.

(a) Show that T is a convex cone and that it is closed w.r.t. τ_f.

(b) Show that for any $p \in \mathbb{R}[x]$ such that $p(x) < 0$ for some $x \in \mathbb{R}$, there exists a hyperplane H such that T lies in the half-space determined by H that does not contain p.

4) Let $\mathbb{R}[x]$ denote the vector space of polynomials in the variable x. Consider the subset $C := \{p \in \mathbb{R}[x] : p = 0 \text{ or } p(x) = \sum_{i=0}^{d} a_i x^i \text{ for some } d \in \mathbb{N}_0, a_0, \dots, a_d \in \mathbb{R} \text{ and } a_d > 0\}$.

(a) Show that C is a convex cone , that $C \cap (-C) = \{0\}$, and that $C \cup (-C) = \mathbb{R}[x]$.

(b) Show that there is no affine hyperplane H such that C is contained only in one of the two half-spaces determined by H.