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TOPOLOGICAL VECTOR SPACES–WS 2018/19

Recap Sheet 1

This recap sheet aims at self-assessing your progress and to recap some of the definitions and concepts introduced in the previous lectures. You do **not** *need to hand in solutions, but please try to answer as many questions as you can since this is a very good training in preparation of your final exam. Exceptionally, this sheet will be discussed on November 9th (Group A) and November 5th (Group B).*

1) Give the definition of basis for a topology on a set and prove the following characterization:

Proposition 1. *Let X be a set and let $\mathcal{B}$ be a collection of subsets of X.*
$\mathcal{B}$ is a basis for a topology τ on X iff the following both hold:

1. *$\mathcal{B}$ covers X, i.e. $\forall x \in X$, $\exists B \in \mathcal{B}$ s.t. $x \in B$. In other words, $X = \cup_{B\in\mathcal{B}} B$.*
2. *If $x \in B_1 \cap B_2$ for some $B_1, B_2 \in \mathcal{B}$, then there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.*

2) Let $\mathcal{B}$ be the collection of all intervals (a, b) in $\mathbb{R}$ together with all the sets of the form $(a, b) - K$, where $K := \{\frac{1}{n} : n \in \mathbb{N}\}$. Prove that $\mathcal{B}$ is the basis for a topology on $\mathbb{R}$, which is usually called the $K-$*topology* on $\mathbb{R}$.

3) Recalling the definitions of filter and basis of a filter on a set, show the following statements.

a) Given a topological space (X, τ) and $x \in X$, the family $\mathcal{F}(x)$ of all neighbourhoods of x is a filter on X. Is every open set a neighbourhood? Is every neighbourhood an open set? Justify your answers!

b) Let $S := \{x_n\}_{n\in\mathbb{N}}$ be a sequence of points in a set X. Then the family $\mathcal{F} := \{A \subset X : |S\backslash A| < \infty\}$ is a filter and it is known as the *filter associated* to the sequence S. For each $m \in \mathbb{N}$, set $S_m := \{x_n \in S : n \geq m\}$. Then $\mathcal{B} := \{S_m : m \in \mathbb{N}\}$ is a basis for $\mathcal{F}$.

4) Establish which of the following topologies on $\mathbb{R}$ are comparable and for each comparable pair say which one is finer.

- τ_1 :=standard topology, whose basis is $\mathcal{B}_1 := \{(a, b) : a, b \in \mathbb{R}$ with $a < b\}$
- $\tau_2 := K-$topology, whose basis $\mathcal{B}_2$ is the one defined in Exercise 1 b)
- $\tau_3 :=$ lower limit topology, whose basis is $\mathcal{B}_3 := \{[a, b) : a, b \in \mathbb{R}$ with $a < b\}$

5) Provide the definitions of closure and interior of a subset A of a topological space (X, τ). Use them to prove that:

a) $x \in \overline{A}$ iff each neighbourhood of x has a nonempty intersection with A.

b) $x \in \mathring{A}$ iff if there exists a neighbourhood of x which entirely lies in A.

c) A is closed in X iff $A = \overline{A}$.

d) A is open in X iff $A = \mathring{A}$.

6) Recall the definition of dense subset of a topological space and give a characterization in terms of open sets. Do you know any example of topological spaces in which every non-empty subset is dense?

7) Recall the definition of convergent sequence in a topological space and show the following statements.

a) Let X be a set endowed with the discrete topology. Then the only convergent sequences in X are the ones that are eventually constant, that is, sequences $\{q_i\}_{i\in\mathbb{N}}$ of points in X such that $q_i = q$ for some $q \in X$ and for all $i \geq N$ for some $N \in \mathbb{N}$.

b) Let Y be a set endowed with the trivial topology. Then every sequence in Y converges to every point of Y.