

# Understanding the Porosity Dependence of Heat Flux Through Glass Fiber Insulation

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## Abstract

In this paper, we describe the mathematical modeling of heat flow across a glass fiber medium. Using different mathematical models we try to explain the porosity dependence of the heat flow which is observed in experiments.

*Key words:* Glass fiber material, heat insulator, porosity depending heat flux, Boussinesq approximation, homogenized models.

## 1 Introduction

Glass fiber insulation plays a significant role in energy conservation in buildings. These insulators are manufactured in the form of thick rectangular sheets and are fitted in the walls (see Figure 1(a) for a cross section through such a sheet). With growing awareness towards energy conservation, in recent time, a lot of research is going on to improve the quality of these insulators. Physical experiments have shown that as the porosity of this insulating material increases, i.e. the amount of glass fibers decreases, the effective heat transfer coefficient of the material also decreases. This is in accordance with the fact that the coefficient of thermal conductivity of glass is quite large compared to that of air. But this phenomenon continues only up to a certain level of porosity. If the porosity crosses that level, the heat transfer coefficient of the insulating material again starts to rise with porosity (see Figure 1(b)) [1, 2]. The main objective of our work is to explain this typical behavior of the heat transfer coefficient by deriving mathematical models for the heat flow through the glass fiber insulation.

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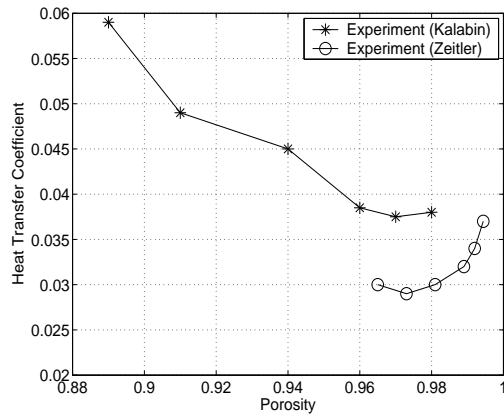
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(a) Glass-wool



(b) Experimental Results

Figure 1: Glass-wool has a very complicated structure and shows an interesting heat flux dependence on porosity.

Due to the complicated microscopic structure of the glass fiber insulation, it is impossible to treat this problem by resolving the heterogeneous domain, so simplifying material models are required. However, one can use the homogenization technique to derive a ‘macroscopic’ model taking into account the ‘microscopic’ heterogeneity. The *Rayleigh number* ( $Ra$ ) plays a very important role in determining the flow regime for such internal flow problems. At low values of  $Ra$  ( $< 10^3$ ), conduction is seen to be the dominant process, however at high values ( $> 10^5$ ) convection dominates. The heat transfer coefficient  $h$  of the material attains its minimum in the transition regime ( $10^3 < Ra < 10^5$ ).

We organize the paper in the following way. In the second section, we present a homogenized model with constant permeability for the problem. It also contains numerical computation related to the model. The ensuing section explores a very simple but useful model to understand the importance of variable fiber density on the heat flow across the medium. We present a modified homogenized model with permeability depending on fiber density and the subsequent numerical results in the fourth section. In the last section we summarize our results.

## 2 Constant Permeability Model

To model a steady heat flow problem in a glass-air domain with moderate temperature differences, a suitable choice is the stationary incompressible Navier Stokes equation with Boussinesq approximation and the stationary convective heat equation for air, combined with stationary heat equation for glass. Here we present the governing equations (nondimensionalized) for the flow, defined in  $\Omega$ , the rectangular two dimensional cross section of the insulator (see Figure 1(a) and Figure 2), which are obtained by periodic homogenization [3, 4, 5] of the afore mentioned equations.

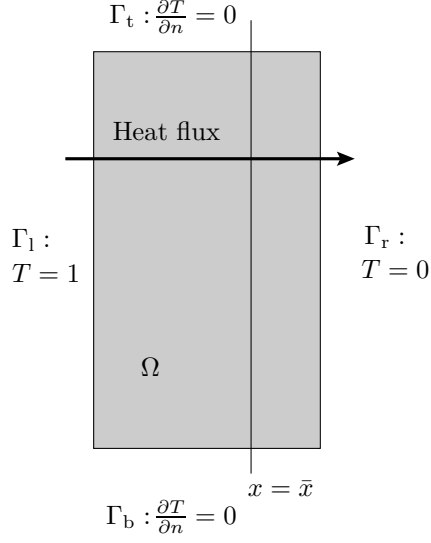


Figure 2: Domain of the homogenized problem with Dirichlet condition at left and right boundary and insulating conditions at top and bottom. The heat flux is computed along a vertical cross section at  $x = \bar{x}$ .

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega \quad (2.1a)$$

$$\mathbf{u} + \nabla p = Ra T \mathbf{e} \quad \text{in } \Omega \quad (2.1b)$$

$$\mathbf{u} \cdot \nabla T = \frac{1}{Pe} \Delta T \quad \text{in } \Omega \quad (2.1c)$$

where  $\mathbf{e} = (0, 1)^t$ ,  $Ra = \frac{k\beta(T_h - T_c)g}{\nu u^*}$  is *Rayleigh number*,  $Pe = \frac{\rho_f c_f u^* L}{\chi_m}$  is *Peclet number*,  $u^*$  and  $L$  are the characteristic velocity and length,  $\beta$  is the coefficient of expansion,  $g$  is the acceleration due to gravity,  $\nu$  is the kinematic viscosity,  $\rho_f$  and  $c_f$  are the density and the coefficient of specific heat of air,  $T_h$  and  $T_c$  are the temperatures at the hot and cold ends,  $\chi_m$  is the thermal conductivity and  $k$  is the permeability of the medium. For permeability at different porosities, we use physically measured values which are given in table 1.

We solve the homogenized equations in  $\Omega$ , a rectangular two dimensional cross section of the insulator, where the normal component of the velocity is zero on the boundary, i.e.  $\mathbf{u} \cdot \mathbf{n} = 0$  on  $\partial\Omega$ . For temperature, we take Dirichlet type boundary conditions  $T_l = 1$  and  $T_r = 0$  on left and right boundaries  $\Gamma_l$  and  $\Gamma_r$  respectively and Neumann type boundary condition  $\frac{\partial T}{\partial \mathbf{n}} = 0$  on top and bottom boundaries  $\Gamma_t$  and  $\Gamma_b$  (see Figure 2).

In order to solve this problem numerically, we require the values of thermal conductivity at different porosities. Obtaining an appropriate thermal conductivity  $\chi_m$  of the homogenized medium is not so straight forward. Nield [6] had shown

that the weighted geometric mean of thermal conductivities of the fluid and solid in a saturated porous medium did serve as a good approximation. For our problem, we approximate it as a convex combination of the weighted arithmetic and harmonic mean of thermal conductivities of glass and air, where we fix the convex combination parameter by using the experimental data.

Now we are mainly interested in finding the average heat flux ( $H$ ) across any vertical cross section  $x = \bar{x}$  in the homogenized domain (see Figure 2), given by,

$$H = \frac{1}{l} \int_0^l \left( (\mathbf{u} \cdot \mathbf{n}) T - \frac{1}{Pe} \frac{\partial T}{\partial x} \right) (\bar{x}, y) dy \quad (2.2)$$

where  $l$  is the length of the domain along the vertical direction. Hence the average heat transfer coefficient  $h$  is given by,

$$h = \frac{H}{T_l - T_r} \quad (2.3)$$

For the numerical solution of the non-linear problem (2.1), we discretize the equations on a fully staggered grid [7] with standard centered differences. We apply a semi-implicit scheme to solve the heat equation and solve a Poisson type equation for the pressure of the fluid flow. Both solvers are combined in an outer iteration to treat the non-linear coupling in the heat equation (for more detailed information, we refer to [8]).

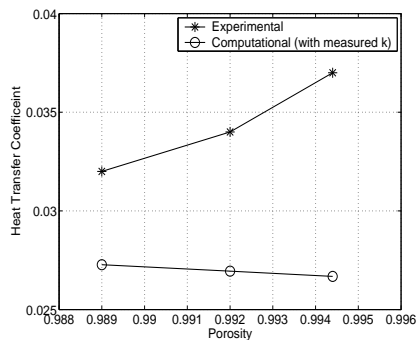


Figure 3: Comparison of Heat Transfer Coefficients

Figure 3 shows a comparison in the heat transfer coefficient  $h$  between physical experiments and computational results with physically measured values of permeability as inputs (see Table 1). From the plot it is clear that the computationally obtained heat transfer coefficients are quite low compared to the experimental values and they are even decreasing instead of increasing so that  $h$  does not reach its minimum in the relevant porosity interval. We have observed a similar behavior of  $h$  for a set of different aspect ratios of the computational domain ranging from 1 to 10. It seems that the permeability is too low to allow a considerable convection process. Now the natural question is: how big must the permeability be to match the experimental data? Going to the

extreme of very high permeability (no glass fibers) we find a heat flux coefficient  $h = 0.1358$  (this value has been obtained by solving the time dependent Navier-Stokes-Boussinesq equation in a rectangular domain of the same shape). Comparing this result with the experimental values in figure 3, we conclude that the glass fibers drastically restrict convection even at a high porosity of 99%. The extreme case of high permeability indicates that a variation of permeability influences the heat transfer considerably. Perhaps, a variation within a reasonable measure tolerance could generate a minimum of  $h$  at the right porosity! This has led our interest in finding the values of permeability required for our model to fit the experiments which can be viewed as an inverse problem. Numerically we treat this as a problem of finding the permeability of the medium which minimizes the difference between computational and experimental value of heat transfer coefficient. The table below shows the experimental (measured) and computational (required) values of permeability.

Porosity	Measured $k$ ( $m^2$ )	Required $k$ ( $m^2$ )
98.9%	$8 \times 10^{-10}$	$6 \times 10^{-8}$
99.2%	$1.15 \times 10^{-9}$	$8 \times 10^{-8}$
99.44%	$1.5 \times 10^{-9}$	$1 \times 10^{-7}$

Table 1: Comparison of Permeability

We understand from the above table that, to match the experimental data, we require the permeability to be two orders of magnitude larger than the measured data. In other words, we cannot match the experimental data by slightly varying measured values taking measurement error into account. Hence we see that the homogenized model with constant permeability model cannot explain the experimental results. This motivates us to modify our existing material model to include macroscopic effects of the material structure (i.e. variation in fiber density).

### 3 Air-Pocket Model

Inspecting figure 1(a), we observe that the fiber density is not uniform throughout the insulating material. In some places the fibers are heavily dense and in some other places they are less dense (i.e. those places contain more air than the rest, called *air-pockets*). In this section, we examine, the influence of these air-pockets on the overall heat flow by considering our flow problem in a very simple geometrical structure and check whether the minimum of  $h$  is attained at the correct porosity range.

In order to arrive at a numerically tractable model, we use a very simple geometry in which air pockets are arranged in a regular way as shown in figure 4. These air pockets are considered to be of the form of hollow rectangular parallelepiped with very thin glass rim, with infinitely extended sides along z-axis in both positive and negative directions. For simplicity all the air-pockets are taken to be of same shape and size. So the porosity of the whole structure is the same as that of a single air-pocket. These cells are described by three parameters :  $d$ ,  $a$ ,  $\theta$  which are shown in Figure 4.

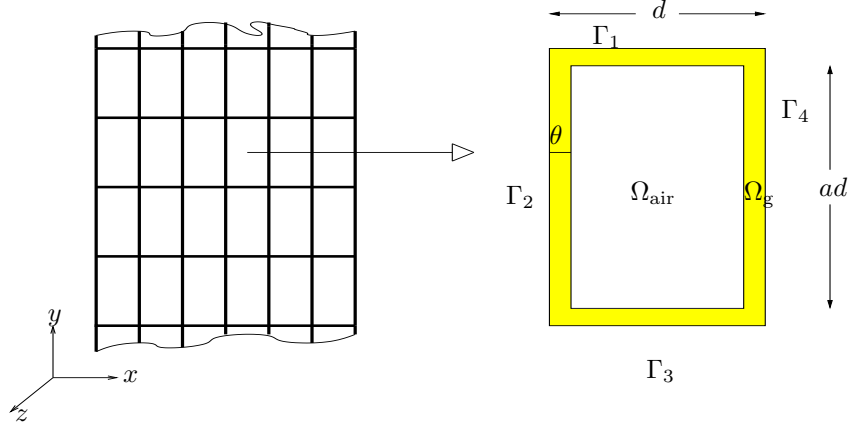


Figure 4: Geometry of Air-Pocket Model

Due to the set up of our problem, it suffices to calculate the heat flux in a cross section of the cell geometry which can be obtained by solving the set of equations described below with suitable boundary conditions. We have scaled the  $y$  direction by  $a$  in order to work with quadratic cells. This leads to the modified gradient,  $\nabla_a = (\frac{\partial}{\partial x}, \frac{1}{a} \frac{\partial}{\partial y})$ .

$$\nabla_a \cdot \mathbf{u} = 0 \quad (3.1a)$$

$$(\mathbf{u} \cdot \nabla_a) \mathbf{u} = c_1 T \mathbf{e} - \nabla_a p + d_1 \nabla_a^2 \mathbf{u} \quad (3.1b)$$

$$\mathbf{u} \cdot \nabla_a T = c_2 \nabla_a^2 T \quad (3.1c)$$

$$k' \nabla_a^2 T = 0 \quad (3.1d)$$

Here  $\mathbf{e} = (0, 1)^t$ ,  $k' = \frac{k_g}{k_{\text{air}}}$ ,  $c_1 = \frac{Gr}{Re^2}$ ,  $c_2 = \frac{1}{RePr}$ ,  $d_1 = \frac{1}{Re}$

At the glass-air interface, we take ‘no slip’ for the velocity of air and continuity of the temperature and the normal heat flux,

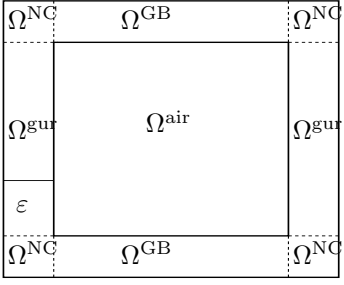
$$\mathbf{u} = 0, \quad T^{\text{air}} = T^{\text{g}}, \quad \frac{\partial T^{\text{air}}}{\partial \mathbf{n}} = k' \frac{\partial T^{\text{g}}}{\partial \mathbf{n}}$$

In an attempt to simplify the model as much as possible, we choose the left and right boundary conditions on temperature in such a way that the problem reduces to a single cell. The trick is *not* to assume Dirichlet boundary conditions at left and right end of the pocket structure but to prescribe only temperature *difference* between the two ends. In our scaling, we assume  $T_l = T_r + 1$  together with equal  $x$ -derivatives of temperature. We have checked that the resulting variation of the temperature at the boundary is below 10% up to porosities of 99.9%. Using periodicity the problem can now be reduced to a single cell problem with boundary conditions,

$$T|_{\Gamma_1} = T|_{\Gamma_3} \quad \text{and} \quad T|_{\Gamma_2} = T|_{\Gamma_4} + \frac{1}{N} \quad (3.3a)$$

$$\left( \frac{\partial T}{\partial y} \right)_{\Gamma_1} = \left( \frac{\partial T}{\partial y} \right)_{\Gamma_3} \quad \text{and} \quad \left( \frac{\partial T}{\partial x} \right)_{\Gamma_2} = \left( \frac{\partial T}{\partial x} \right)_{\Gamma_4} \quad (3.3b)$$

where  $N$  is the number of cells in a row of the model geometry. Even the numerical treatment of a single air-pocket leads to the problem that, as the porosity increases, the thickness of the glass rim decreases. So at high porosity levels this rules out a uniform regular meshing of the domain. This is particularly unwanted as the contribution to heat flux due to these thin rims decreases proportionally to rim thickness (if temperature gradients are bounded). This suggests the use of an asymptotic approximation of the rim influence. The approximation is based on the following expansions of temperature, pressure and velocity for small  $\varepsilon$ , where  $\varepsilon$  is the scaled rim thickness  $\frac{\theta}{d}$  and  $(x, y)$  are local coordinates in the rectangular regions shown in the next figure.

$$\begin{aligned}
T^{\text{GB}}(x, y) &= \sum_{i=0}^{\infty} \varepsilon^i T_i^{\text{g}} \left( \frac{x}{1-2\varepsilon}, \frac{y}{\varepsilon} \right) \\
T^{\text{gur}}(x, y) &= \sum_{i=0}^{\infty} \varepsilon^i T_i^{\text{g}} \left( \frac{x}{\varepsilon}, \frac{y}{1-2\varepsilon} \right) \\
T^{\text{NC}}(x, y) &= \sum_{i=0}^{\infty} \varepsilon^i T_i^{\text{g}} \left( \frac{x}{\varepsilon}, \frac{y}{\varepsilon} \right) \\
T^{\text{air}}(x, y) &= \sum_{i=0}^{\infty} \varepsilon^i T_i^{\text{air}} \left( \frac{x}{1-2\varepsilon}, \frac{y}{1-2\varepsilon} \right)
\end{aligned}$$


The functions  $\mathbf{u}^{\text{air}}$  and  $p^{\text{air}}$  are expanded similar to  $T^{\text{air}}$ . Carrying out the analysis, we find partial differential equations for the air quantities on the unit square while the glass quantities eventually give rise to the boundary conditions for the air part. In particular, the zeroth order air quantities solve the equations (3.1) with boundary conditions (3.3) in the unit square and the first order quantities satisfy the linear problem,

$$\begin{aligned}
\nabla_a \cdot \mathbf{u}_1 &= 0 \\
(\mathbf{u}_1 \cdot \nabla_a) \mathbf{u}_0 + (\mathbf{u}_0 \cdot \nabla_a) \mathbf{u}_1 - d_1 \Delta_a (2\mathbf{u}_0 + \mathbf{u}_1) &= c_1 \mathbf{e} (T_1^{\text{air}} - 2T_0^{\text{air}}) - \nabla_a p_1 \\
-2c_2 \Delta_a T_0^{\text{air}} + \mathbf{u}_1 \cdot \nabla_a T_0^{\text{air}} + \mathbf{u}_0 \cdot \nabla_a T_1^{\text{air}} &= c_2 \Delta_a T_1^{\text{air}}
\end{aligned}$$

Now the boundary conditions generated by the glass part are ‘no slip’ condition for the velocity i.e.  $\mathbf{u}_1 = 0$  and periodic conditions (with prescribed jump) for the temperature,

$$\begin{aligned}
T_1^{\text{air}}(x, 0) - T_1^{\text{air}}(x, 1) &= \frac{2}{k'} \frac{\partial T_0^{\text{air}}}{\partial y}(x, 1) & x \in [0, 1] \\
T_1^{\text{air}}(0, y) - T_1^{\text{air}}(1, y) &= \frac{2}{k'} \frac{\partial T_0^{\text{air}}}{\partial x}(1, y) & y \in [0, 1] \\
\frac{\partial T_1^{\text{air}}}{\partial x}(0, y) - \frac{\partial T_1^{\text{air}}}{\partial x}(1, y) &= -\frac{k'}{a^2} \frac{\partial^2}{\partial y^2} \left[ T_0^{\text{air}}(0, y) + T_0^{\text{air}}(1, y) \right] & y \in [0, 1] \\
\frac{\partial T_1^{\text{air}}}{\partial y}(x, 0) - \frac{\partial T_1^{\text{air}}}{\partial y}(x, 1) &= -k' a^2 \frac{\partial^2}{\partial x^2} \left[ T_0^{\text{air}}(x, 0) + T_0^{\text{air}}(x, 1) \right] & x \in [0, 1]
\end{aligned}$$

The glass quantities are then given explicitly as linear functions with coefficients determined by the boundary values of the air quantities.

To solve our problem numerically, a semi-implicit scheme is used for heat equation and Uzawa algorithm [9] (with the nonlinear part of Navier Stokes equation

along with the force term as the right hand side in the case of the leading order problem) is employed for the flow part. The discretization is again carried out on a staggered grid with standard differences as described in the previous section. Finally, the model parameters to fit the physical experiments are found by minimizing the function  $E$ ,

$$E(a, \theta) = \sum_i (w_i (h_{cp}(a, \theta, \phi_i) - h_{ex}(\phi_i)))^2$$

where  $h_{cp}(a, \theta, \phi_i)$  and  $h_{ex}(\phi_i)$  are values of heat transfer coefficient obtained by computations and experiments respectively,  $w_i$  are weights and  $\phi_i$  are the porosities at which experimental data are available.

The following plots compare the experimental data with computational results taking two different sets of weights ( $[10 \ 0.1 \ 0.5 \ 30 \ 0.5 \ 40]$  and  $[20 \ 0.1 \ 5 \ 40 \ 10 \ 40]$  for Kalabin’s experimental data,  $[10 \ 10 \ 10 \ 1 \ 40 \ 3]$  and  $[1 \ 50 \ 50 \ 10 \ 40 \ 2]$  for Zeitler’s experimental data respectively) in each case.

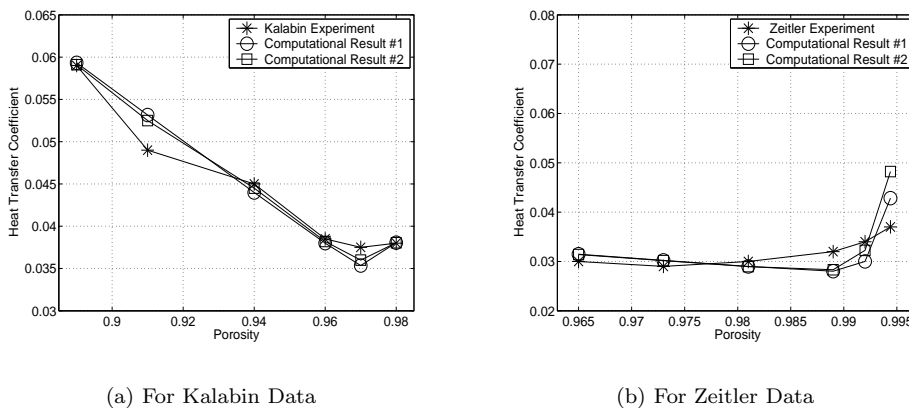


Figure 5: Comparison of Results

In the above plots we can observe that it is possible to find parameters for our mathematical model to reproduce a qualitatively similar behavior of the heat transfer coefficient as that of experiments. After analyzing the computationally obtained data sets for the model parameters, we see that a typical parameter set corresponding to ‘Kalabin experiment’ indicates that at 98% porosity, the typical dimension of an air-pocket is approximately  $1.5 \times 2.7 \text{ cm}^2$  and that of ‘Zeitler experiment’ indicates that at 99.2% porosity, the typical dimension of an air-pocket is approximately  $1.3 \times 3.5 \text{ cm}^2$ . In reality too we can find low density regions of similar size and shape, or in other words computationally obtained air-pockets are of “reasonable shape”. Hence we can infer that the Air-pocket model may reasonably predict the minimum of  $h$  in the experimentally observed porosity range.



## 4 Model with Variable Permeability

While the previous model shows the importance of isolated air-pockets for the overall heat flux it neglects two aspects observed in the fiber material: first, true air-pockets of reasonable size are very rare but regions of higher permeability exist (see Figure 1(a)). Second, the air-pocket model only allows local flows while an overall circulation is actually possible in fiber material. To refine the air-pocket approach we therefore consider a 2D homogenized model which includes a spatial variation of the permeability. Specifically, air-pockets are replaced with localized regions of higher permeability and regions of low permeability generalize the glass rims of the air-pocket model. The advantage of the variable permeability approach is that the pockets need not be arranged regularly and may be of different size which is more realistic. Moreover, the regions of low permeability (the rims) do not block the flow completely but allow for an overall circulation which reflects a property of the fiber material. Our goal is to show that the permeability distribution can be chosen in such a way that both the experimentally found heat transfer values and the experimental value of the average permeability can be recovered.

The assumption of variable permeability leads to equations (2.1) with (2.1b) of the form,

$$\mathbf{u} + K' \nabla p = Ra T \mathbf{e} \quad (4.1)$$

where  $K' = \frac{K}{k}$  and  $k$  is a characteristic value of the permeability.

At variance with common variable permeability models, which are based on exponentially decaying functions [10], to describe the permeability we choose the function  $K$  in such a way that it describes high, medium and low fiber density in the insulator. Taking  $k_1$  and  $k_2$  respectively as the permeability at the high and the low density regions, we define,

$$\psi(x, y) = k_1 + (k_2 - k_1) \sin^2(\pi x) \sin^2(\pi y), \quad (x, y) \in [0, 1]^2$$

and subsequently define,

$$K(x, y) = \sum_{i=1}^M \psi(\bar{a}_i(x - \bar{x}_i), \bar{b}_i(y - \bar{y}_i))$$

where  $M$  represents the number of low density regions in the domain. Now we look for suitable values of  $k_1$  and  $k_2$  to fit the experiments by treating this problem as an inverse problem. For our simulation, we have considered six low density regions which are randomly located in the domain and choice of their dimensions is motivated by the results of air-pocket model. The following table presents the values of  $k_1$ ,  $k_2$  which are needed to get the experimentally found heat transfer coefficients. By an additional numerical flow experiment, we also determine the corresponding average permeability of the model. It turns out to be quite close to the measured value.

$\phi$	$k_1 (m^2)$	$k_2 (m^2)$	Average $k(m^2)$	Measured $k(m^2)$
98.9%	$1.7 \times 10^{-10}$	$1.615 \times 10^{-06}$	$1.482 \times 10^{-09}$	$8 \times 10^{-10}$
99.2%	$6.3 \times 10^{-10}$	$2.027 \times 10^{-06}$	$5.163 \times 10^{-09}$	$1.15 \times 10^{-09}$
99.44%	$1.0 \times 10^{-09}$	$2.605 \times 10^{-06}$	$8.045 \times 10^{-09}$	$1.5 \times 10^{-09}$

Table 2 : Comparison of Permeability

## 5 Summary

We have started with a constant permeability homogenized model to describe the heat flow in the medium and have shown that this model is not accurate enough to reflect the experimental findings. On the other hand, a very simple air-pocket model suffices to reproduce the results which highlights the importance of macroscopic permeability variations. We have shown that a refined model can also reproduce the results with properly defined permeabilities. Our observations indicate that the quality of the glass fiber insulators can be improved by reducing the inhomogeneities in the glass fiber density as much as possible. To refine and validate the models further, additional experiments are required which allow to quantify the inhomogeneities in the fiber density.

### Acknowledgment

One of the authors, Samir Roy, would like to thank German Academic Exchange Service (DAAD) for the financial support when the work was carried out at TU-KL, Germany and Council of Scientific and Industrial Research (CSIR) India for the financial support (JRF) at IIT Madras.

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