Mixed order systems on spaces of mixed scales and application to N-parabolic boundary value problems

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Introduction and motivation

In this work we want to present a theory for mixed order systems which are realized on spaces of mixed scales of Sobolev and Besov spaces. For an example of a mixed order system we introduce the **Stokes problem on $\mathbb{R}^{\mathbf{n}}$** which reads as

$$\begin{cases} \partial_t u - \Delta u + \nabla\pi = f, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n, \\ \operatorname{div} u = g, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n, \\ u_{|t=0} = 0 \end{cases} \tag{1}$$

for the unknown functions $u : \mathbb{R}^n \to \mathbb{R}^n$ and $\pi : \mathbb{R}^n \to \mathbb{R}$. After a formal Fourier/Laplace transform and a complex extension to (bi)sectors we obtain

$$\begin{pmatrix} (\partial_t - \Delta)\mathrm{id}_n & \nabla \\ \nabla^T & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} (\lambda + |z|_-^2)\mathrm{id}_n & z \\ z^T & 0 \end{pmatrix} = \begin{pmatrix} \mathrm{id}_n & 0 \\ 0 & |z|_- \end{pmatrix} \cdot \underbrace{\begin{pmatrix} (\lambda + |z|_-^2)\mathrm{id}_n & |z|_-^{-1} z \\ |z|_-^{-1} z^T & 0 \end{pmatrix}}_{=:\mathscr{L}(\lambda,z)} \cdot \begin{pmatrix} \mathrm{id}_n & 0 \\ 0 & |z|_- \end{pmatrix}$$

with $(\lambda, z) \in S_\theta \times \Sigma_\delta^n$, $|z|_- := \sqrt{-\sum_{k=1}^n z_k^2}$. Due to the fact that the symbols in $\mathscr{L}$ do not represent differential operators we first have to provide a suitable functional calculus. After an analysis of the order structure of L we can give spaces such that $\mathscr{L}(\nabla_+)$ acts as an isomorphism between them.
Mixed order systems also occur at the treatment of **parabolic boundary value problems** when it is **reduced to the boundary**. Due to this reduction we have to consider mixed order systems on trace spaces where mixed scales as ${}_0B_p^s(\mathbb{R}_+, H_p^r(\mathbb{R}^n))$ and ${}_0H_p^s(\mathbb{R}_+, B_p^r(\mathbb{R}^n))$ naturally appear.

Holomorphic functional calculus for (bi)sectorial operators

Let $\mathbf{T} := (T_1, \ldots, T_N)$ be a tuple of sectorial or bisectorial operators (i.e. $\sigma(T_k) \subseteq \overline{S_\theta}$ or $\sigma(T_k) \subseteq \overline{\Sigma_\delta}$ plus resolvent estimate) on the Banach space X. For an open set $\Omega := \prod_{k=1}^N \Omega_k \subset \mathbb{C}^N$ (Ω_k sector or bisector) we define the sets of holomorphic functions

$$H_0^\infty(\Omega) := \left\{ f \in H(\Omega, X) : \exists C, s > 0 \forall z \in \Omega : \|f(z)\|_Y \le C \prod_{k=1}^N \min\{|z_k|^s, |z_k|^{-s}\} \right\},$$

$$H_P(\Omega) := \left\{ f \in H(\Omega, X) : \exists C > 0, s \in \mathbb{R} \forall z \in \Omega : \|f(z)\|_Y \le C \prod_{k=1}^N \max\{|z_k|^s, |z_k|^{-s}\} \right\},$$

and $H^\infty(\Omega)$ the set of holomorphic and bounded functions. At first we define the operator

$$f(\mathbf{T}) := \frac{1}{(2\pi i)^N} \int_\Gamma f(z) \prod_{k=1}^N (z_k - T_k)^{-1} dz_{[L(X)]} \in L(X), \quad f \in H_0^\infty(\Omega)$$

for a suitable path Γ. This calculus can then be extended to $f \in H_P(\Omega)$ by

$$f(\mathbf{T})x := \Psi(\mathbf{T})^{-m}(\Psi^m f)(\mathbf{T})x, \quad x \in D(f(\mathbf{T})) := \{x \in X : (\Psi^m f)(\mathbf{T})x \in R(\Psi(\mathbf{T})^m)\}$$

where Ψ is a certain shift function such that there exists $m \in \mathbb{N}$ with $\Psi^m f \in H_0^\infty(\Omega)$. The tuple $\mathbf{T}$ admits a **bounded joint H^∞-calculus** if there exists $C > 0$ such that

$$\|f(\mathbf{T})\|_{L(X)} \le C\|f\|_\infty \text{ for all } f \in H^\infty(\Omega).$$

For more details we refer to [5].

Joint functional calculus for $\nabla_+ := (\partial_t, \partial_1, \ldots, \partial_n)$

We consider realizations of the derivative operators $(\partial_t, \partial_1, \ldots, \partial_n)$ on spaces of mixed scales. For $\mathcal{W} := {}_0\mathcal{F}^s(\mathbb{R}_+, \mathcal{K}^r(\mathbb{R}^n))$ with $s, r \ge 0$, $p_0, p_1, q_0, q_1 \in (0, \infty)$, $\mathcal{F} \in \{H_{p_0}, B_{p_0 q_0}\}$, and $\mathcal{K} \in \{H_{p_1}, B_{p_1 q_1}\}$ the operators

$$\partial_k^{\mathcal{W}} : D\left(\partial_k^{\mathcal{W}}\right) \subset \mathcal{W} \to \mathcal{W}, \quad u \mapsto \partial_k u, \qquad D\left(\partial_k^{\mathcal{W}}\right) := {}_0\mathcal{F}^s(\mathbb{R}_+, \mathcal{K}^{r+1}(\mathbb{R}^n)), \quad k = 1, \ldots, n$$
$$\partial_t^{\mathcal{W}} : D\left(\partial_t^{\mathcal{W}}\right) \subset \mathcal{W} \to \mathcal{W}, \quad u \mapsto \partial_t u, \qquad D\left(\partial_k^{\mathcal{W}}\right) := {}_0\mathcal{F}^{s+1}(\mathbb{R}_+, \mathcal{K}^r(\mathbb{R}^n))$$

are bisectorial, respectively sectorial. The authors of [4] showed that $(\partial_1, \ldots, \partial_n)$ admits a joint H^∞-calculus on $L^p(\mathbb{R}_+^n)$ with

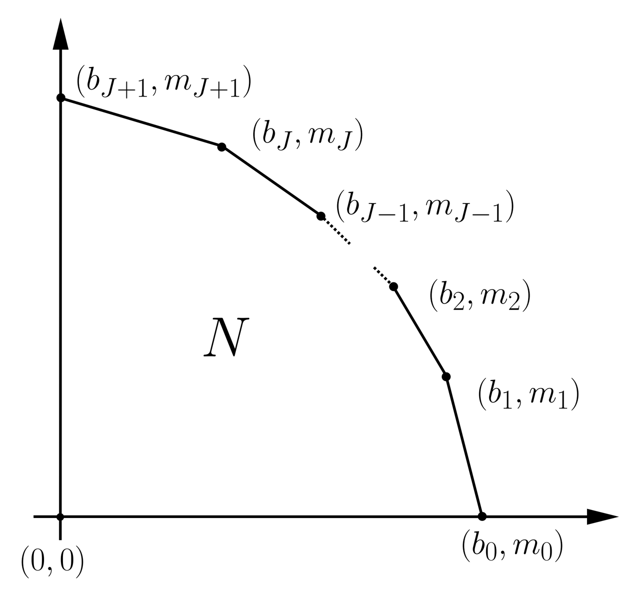
$$f(\partial_1, \ldots, \partial_n) = \mathscr{F}^{-1} f(i\cdot)\mathscr{F} \in L(L^p(\mathbb{R}_+^n)), \quad f \in H^\infty(\Omega).$$

Using common results for isomorphisms and real interpolation this result can be generalized to the ground space $\mathcal{W}$. Due to the fact that ∂_t even admits an **$\mathcal{R}$-bounded H^∞-calculus** we can apply the **Kalton-Weis Theorem** in a version of G. Dore and A. Venni (c.f. [5]) for tuples of bisectorial operators and obtain that $\nabla_+^{\mathcal{W}} := (\partial_{\mathbf{t}}^{\mathcal{W}}, \partial_{\mathbf{1}}^{\mathcal{W}}, \ldots, \partial_{\mathbf{n}}^{\mathcal{W}})$ **admits a bounded joint H^∞-calculus** which also can be represented by means of the Fourier transform.

Order structure, Newton polygon, and N-parabolicity

One problem of the holomorphic functional calculus from above is the definition of the domain of $f(\partial_1, \ldots, \partial_n)$ which is very unconstructive. For the moment we can not argue that $\Lambda(\partial_1, \ldots, \partial_n) = \Delta$ (with $\Lambda(z) := \sum_{k=1}^n z_k^2$) which especially involves $D(\Lambda(\partial_1, \ldots, \partial_n)) = W_p^2(\mathbb{R}^n)$. To derive more information about the domain $D(f(\nabla_+))$ it is convenient to ask for **estimates by weight functions** related to a Newton polygon $N = \mathrm{conv}(N_c)$ with $N_c := \{(0,0)\} \cup \{(b_\ell, m_\ell) : \ell = 0, \ldots, J\}$ i.e.

$$|f(\lambda, z)| \le C \cdot \sum_{(\alpha,\beta) \in N_c} |z|^\alpha |\lambda|^\beta. \tag{2}$$



If f has this **'upper order structure'** we can already conclude

$$\bigcap_{(\alpha,\beta) \in N_c} {}_0\mathcal{F}^{s+\beta}(\mathbb{R}_+, \mathcal{K}^{r+\alpha}(\mathbb{R}^n)) \subseteq D(f(\nabla_+^{\mathcal{W}})).$$

For the handling of quotients of holomorphic functions we sometimes need the converse estimate

$$\sum_{(\alpha,\beta) \in N_c} |z|^\alpha |\lambda|^\beta \le C \cdot |f(\lambda, z)|. \tag{3}$$

The function f is called **N-parabolic** if it has this **'lower order structure'**. For an equivalent characterization of N-parabolic symbols we continued the work of L.R. Volevich, S. Gindikin, R. Denk, J. Saal, and J. Seiler ([2], [3], [6]) about **non-vanishing $\boldsymbol{\gamma}$-principal parts**. A function $\mathcal{O}(\gamma) := \max_\ell\{\gamma \cdot m_\ell + b_\ell\}$ ($\gamma > 0$) such that $\{(b_\ell, m_\ell)\}_\ell$ are the corners of a Newton polygon is called **upper/lower order function** for $f \in H_P(\Omega)$ if estimate (2) resp. (3) holds.

Main Theorem for mixed order systems

We consider a **totally non-degenerated Douglis-Nirenberg system** $\mathscr{L} \in [H_P(\Omega)]^{m \times m}$ (i.e. there are order functions $s_j(\gamma) = \max_\ell\{\gamma \cdot m_\ell(s_j) + b_\ell(s_j)\}, t_i(\gamma) = \max_\ell\{\gamma \cdot m_\ell(t_i) + b_\ell(t_i)\}$ such that $s_j + t_i$ is an upper order function for $\mathscr{L}_{ji}$ and $\sum_{j=1}^m (s_j + t_j)$ is a lower and upper order function of $\det \mathscr{L}$).

Theorem. *Let $s'_\ell \ge \max_j\{m_\ell(s_j)\}$, $r'_\ell \ge 0$, and $\mathcal{F}_\ell \in \{H_{p_0}, B_{p_0 q_0}\}$, $\mathcal{K}_\ell \in \{H_{p_1}, B_{p_1 q_1}\}$. If certain embeddings hold then there exists $\varrho_0 > 0$ such that for all $\varrho \ge \varrho_0$*

$$[\mathscr{L}_\varrho(\nabla_+)]_{|\mathbb{H}} \in L_{\mathrm{Isom}}(\mathbb{H}, \mathbb{F}), \qquad ([\mathscr{L}_\varrho(\nabla_+)]_{|\mathbb{H}})^{-1} = [\mathscr{L}_\varrho^{-1}(\nabla_+)]_{|\mathbb{F}}$$

with $\mathscr{L}_\varrho(\lambda, z) := \mathscr{L}(\varrho + \lambda, z)$, $\mathbb{H} := \prod_{i=1}^m \mathbb{H}_i$, $\mathbb{F} := \prod_{i=1}^m \mathbb{F}_i$, and

$$\mathbb{H}_i := \bigcap_{\ell=0}^J {}_0\mathcal{F}_\ell^{s'_\ell + m_\ell(t_i)}\left(\mathbb{R}_+, \mathcal{K}_\ell^{r'_\ell + b_\ell(t_i)}(\mathbb{R}^n)\right), \qquad \mathbb{F}_j := \bigcap_{\ell=0}^J {}_0\mathcal{F}_\ell^{s'_\ell - m_\ell(s_j)}\left(\mathbb{R}_+, \mathcal{K}_\ell^{r'_\ell - b_\ell(s_j)}(\mathbb{R}^n)\right).$$

Remark.

(I) The ϱ-shift of the system can be removed by the implementation of exponential weights in the time variable.

(II) This result can be generalized to negative order functions. This is necessary for the handling of the L_p-L_q Stokes problem on $\mathbb{R}^n$.

(III) For the embeddings mentioned in the assumption we need a kind of 'compatibility' for the used scale $(\mathcal{F}_\ell, \mathcal{K}_\ell)_\ell$. The scale $(\mathcal{F}_\ell, \mathcal{K}_\ell) \in \{(B_{p_0}, H_{p_1}), (H_{p_0}, B_{p_1 p_0})\}$ is tame in this context for example.

Applications

(I) **Maximal $\mathbf{L_p}$-$\mathbf{L_q}$-regularity for the Stokes system on $\mathbb{R}^{\mathbf{n}}$:** Using the notation as in the introduction our results yield $[\mathscr{L}(\nabla_+)]_{|\mathbb{H}} \in L_{\mathrm{Isom}}(\mathbb{H}, \mathbb{F})$ for

$$\mathbb{H} := \left[{}_0H_{p,\varrho}^1(\mathbb{R}_+, L_q(\mathbb{R}^n)) \cap L_{p,\varrho}(\mathbb{R}_+, H_q^2(\mathbb{R}^n))\right]^n \times L_{p,\varrho}(\mathbb{R}_+, L_q(\mathbb{R}^n))$$
$$\mathbb{F} := \left[L_{p,\varrho}(\mathbb{R}_+, L_q(\mathbb{R}^n))\right]^n \times \left({}_0H_{p,\varrho}^1(\mathbb{R}_+, L_q(\mathbb{R}^n)) \cap L_{p,\varrho}(\mathbb{R}_+, H_q^2(\mathbb{R}^n))\right),$$

and $\varrho \ge \varrho_0$. Due to the fact that $|z|_-$ shifts into homogenous Sobolev spaces we already obtain

$$\begin{pmatrix} (\partial_t - \Delta)\mathrm{id}_n & \nabla \\ \nabla^T & 0 \end{pmatrix} \in L_{\mathrm{Isom}}\Big(\left[{}_0H_{p,\varrho}^1(\mathbb{R}_+, L_q(\mathbb{R}^n)) \cap L_{p,\varrho}(\mathbb{R}_+, H_q^2(\mathbb{R}^n))\right]^n \times L_{p,\varrho}(\mathbb{R}_+, \dot{H}_q^1(\mathbb{R}^n)),$$
$$\left[L_{p,\varrho}(\mathbb{R}_+, L_q(\mathbb{R}^n))\right]^n \times \left({}_0H_{p,\varrho}^1(\mathbb{R}_+, \dot{H}_q^{-1}(\mathbb{R}^n)) \cap L_{p,\varrho}(\mathbb{R}_+, H_q^1(\mathbb{R}^n))\right)\Big).$$

(II) **N-Parabolic boundary value problems :** After a formal Laplace/Fourier transform in the time/space variables (i.e. $(t, x') \rightsquigarrow (\lambda, \xi')$) the boundary value problem reduces to an ODE in the x_n-variable. Using the remainders of the symbols of the boundary operators after a certain modulo operation we derive the **Lopatinskii-matrix** L. This matrix is the connection between the original problem and the associated Dirichlet problem. Indeed after plugging in $\nabla_+ := (\partial_t, \partial_1, \ldots, \partial_n)$ the operator $L(\nabla_+)$ acts as a **generalized Dirichlet to Neumann operator**.

References

[1] H. Amann. Anisotropic function spaces and maximal regularity for parabolic problems. part 1: Function spaces. *Jindrich Necas Center for Mathematical Modeling Lecture Notes*, 6, 2009.

[2] R. Denk, J. Saal, and J. Seiler. Inhomogeneous symbols, the Newton polygon, and maximal L^p-regularity. *Russian J. Math. Phys*, 15(2):171–192, 2008.

[3] R. Denk and L.R. Volevich. Parabolic boundary value problems connected with the newtons's polygon and some problems of crystallization. *J. Evol. Equ.*, 8:523–556, 2008.

[4] G. Dore and A. Venni. H^∞-calculus for an elliptic operator on a halfspace with general boundary conditions. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5)*, 1(3):487–543, 2002.

[5] G. Dore and A. Venni. H^∞ functional calculus for sectorial and bisectorial operators. *Studia Math.*, 166(3):221–241, 2005.

[6] S. Gindikin and L. R. Volevich. *The method of Newton's polyhedron in the theory of partial differential equations*, volume 86 of *Mathematics and its Applications (Soviet Series)*. Kluwer Academic Publishers Group, Dordrecht, 1992. Translated from the Russian manuscript by V. M. Volosov.

[7] P.C. Kunstmann and L. Weis. *Maximal L_p-regularity for Parabolic Equations, Fourier Multiplier Theorems, and H^∞-functional Calculus*, volume 1855 of *Functional Analytic Methods for Evolution Equations, Lecture notes in Math.* Springer, 2004.