On the Spectral Stability of Internal Solitary Waves in Stratified Fluids

Andreas Klaiber (University of Konstanz)

1. INTRODUCTION

(a) Channel model for internal waves

Euler equations:

$$\begin{aligned} u_x + v_y &= 0 \\ \rho_t + u\rho_x + v\rho_y &= 0 \\ \rho\,(u_t + uu_x + vu_y) &= -p_x \\ \rho\,(v_t + uv_x + vv_y) &= -p_y - g\rho \end{aligned}$$

Domain: $\{(x,y) : x \in \mathbb{R}, 0 < y < 1\} \subset \mathbb{R}^2$

Boundary conditions: $v(t,x,y=0) = v(t,x,y=1) = 0$

Assumptions on the fluid:

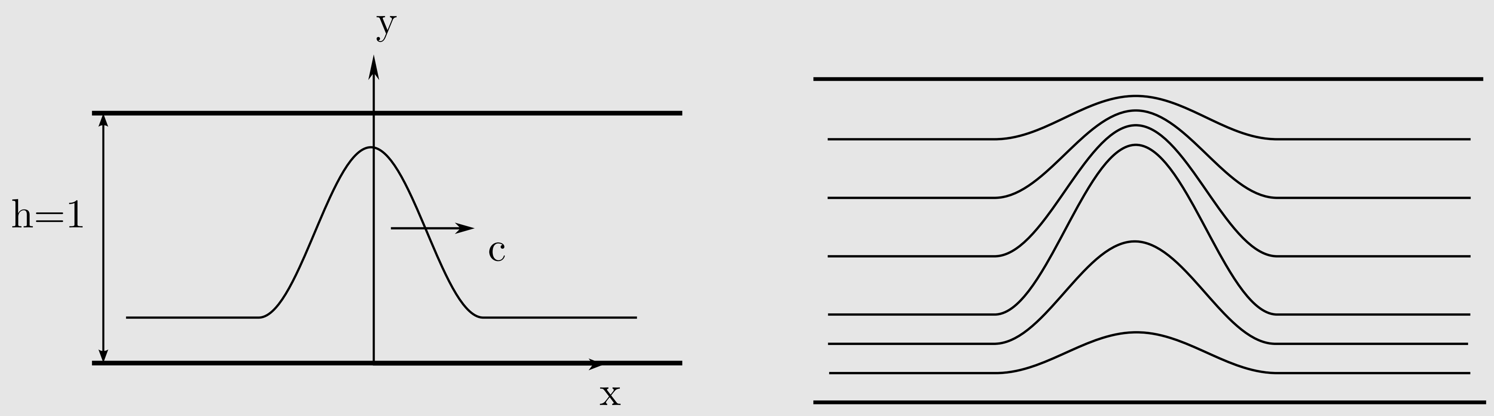
- incompressible, inviscid
- inhomogeneous density, at rest given by $\bar\rho(y) = \mathrm{e}^{-\delta y}$

(b) Existence of internal solitary waves

An Internal Solitary Wave (ISW) of speed c is a solution of the form

$$U(t,x,y) = U^c(x - ct, y)$$

with a travelling-wave profile U^c which decays uniformly to the quiescent state $\bar U(y) = (\bar\rho(y), 0, 0, \bar p(y))$.



For proofs of the existence of ISWs, see e. g.: [BBT, LF]

2. GOAL AND STRATEGY

(a) Goal

(Q1) How to detect analytically (in-)stability of ISWs? Stability in which sense?

(Q2) Are small-amplitude ISWs stable like their leading order parts, the Korteweg-deVries solitons?

(b) Strategy

Starting from Euler eigenvalue problem associated with ISW $U^c(\xi, y)$, Evans function approach to spectral stability consisting of 5 steps:

1. spatial-dynamics formulation (EVP) of eigenvalue problem
2. Galerkin-type truncation procedure to obtain finite-dimensional problems (EVP$_N$), $N \in \mathbb{N}$
3. definition of an Evans function D_N for each truncated problem
4. investigation of D_N for zeros κ with $\Re\kappa > 0$
5. implications of Step 4 for (EVP) and original eigenvalue problem

3. MAIN RESULTS

Thm. 1. The eigenvalue problem can be written as the abstract ODE

$$W'(\xi) = \mathbb{A}(\xi; \kappa, c)W(\xi) \qquad \text{(EVP)}$$

in the space $\mathcal{W} = \left(L^2(0,1)\right)^4$, in a formal sense.

Thm. 2. With respect to a suitable Hilbert basis and for any $N \in \mathbb{N}$, (EVP) has a finite-dimensional truncation

$$w'(\xi) = \mathcal{A}(\xi; \kappa, c, N)w(\xi) \qquad (\mathrm{EVP}_N)$$

in the space $\mathbb{C}^{4(N+1)}$ with $\mathcal{A} = \mathcal{A}^\infty(\kappa, c) + \mathcal{B}(\xi; \kappa, c)$, $\mathcal{B}(\pm\infty; \kappa, c) = 0$.

Thm. 3. For any $N \in \mathbb{N}$ there is an analytic Evans function $D_N : \Omega \supset \overline{\mathbb{C}_+} \to \mathbb{C}$ with the properties

(a) $D_N(\kappa) = 0$ with $\Re\kappa > 0$ iff (EVP_N) has a solution with $w(\pm\infty) = 0$,

(b) $D_N(0) = 0$ and $D_N'(0) = 0$.

Thm. 4. For small-amplitude ISWs, i.e. $c - c_0 = \varepsilon^2 \ll 1$, and for any $R > 0$ the Evans function satisfies

$$D_N(\kappa) \neq 0 \quad \text{for all } \kappa \neq 0 \text{ with } 0 \leq |\kappa| \leq R\varepsilon^3.$$

Idea of proof. (i) Approximate expression for small ISWs known (see [Be, Ki, Ja]),

$$\rho^c(\xi, y) = \bar\rho(y) + a_\varepsilon(\xi)\Phi(y) + O(|a_\varepsilon|^2),$$

where $a_\varepsilon(\xi) = \varepsilon^2 A_*(\varepsilon\xi)$ + h.o.t. with a Korteweg-deVries soliton A_*.
$\rightsquigarrow$ slow-fast structure of (EVP$_N$).

(ii) Rescaled reduced problem contains KdV eigenvalue problem associated with soliton A_* PLUS small perturbations.

(iii) Spectral stability of KdV soliton A_* ([PW] via D_{KdV}) implies absence of zeros for D_N in the regime $\kappa = O(\varepsilon^3)$.

4. DIRECTIONS OF RESEARCH

- ○ complete treatment of (EVP$_N$) for the small-amplitude regime
- ○ convergence of the sequence $(D_N)_{N\in\mathbb{N}}$ of Evans functions
- ○ rigorous functional-analytic setting for Steps 1 and 2
- ○ investigation of Step 5
- ○ generalization to more general stratifications $\bar\rho$

Acknowledgment. I have obtained the presented results during my PhD studies with Prof. Heinrich Freistühler at University of Konstanz, see [Kl].

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Andreas Klaiber • University of Konstanz, Germany • Andreas.Klaiber@uni-konstanz.de