Viscous Profiles for Shock Waves in Isentropic Magnetohydrodynamics

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ISENTROPIC MHD IN 1D

Equations:

$$\begin{aligned} \partial_t\rho + \partial_x(\rho v) &= 0 \\ \partial_t(\rho v) + \partial_x\left(\rho v^2 + \pi(\rho) + \frac{1}{2}|\mathbf{b}|^2\right) &= \mu\partial_{xx}v \\ \partial_t(\rho\mathbf{w}) + \partial_x(\rho v\mathbf{w} - a\mathbf{b}) &= \nu\partial_{xx}\mathbf{w} \\ \partial_t\mathbf{b} + \partial_x(v\mathbf{b} - a\mathbf{w}) &= \eta\partial_{xx}\mathbf{b} \end{aligned} \qquad (\mathrm{PDE})$$

Assumptions:

- ∘ positive viscosity coefficients $\mu - \nu > 0$, $\nu > 0$, $\eta > 0$
- ∘ general barotropic pressure law $\pi(\rho)$ with

$$\pi(\rho) > 0, \pi'(\rho) > 0, \pi''(\rho) \geq 0, \lim_{\rho\to 0}\pi(\rho) = 0, \lim_{\rho\to\infty}\pi(\rho) = \infty$$

Properties of *inviscid* part, i.e. $\mu = \nu = \eta = 0$:

- ∘ non-strictly hyperbolic system
- ∘ allows for various shock wave solutions

VISCOUS PROFILES FOR SHOCK WAVES

A viscous profile for the shock wave

$$U^s(x,t) = \begin{cases} U^-, & x < st \\ U^+, & x > st \end{cases}$$

($U^\pm$ related by Rankine-Hugoniot conditions) is a solution $\Phi(x - st)$ of (PDE) with the same asymptotic states: $\lim_{\xi\to\pm\infty}\Phi(\xi) = U^\pm$. Φ corresponds (for $s = 0$ w.l.o.g) to a heteroclinic orbit of:

$$\begin{aligned} \mu v' &= mv + p(v) + \frac{1}{2}|\mathbf{b}|^2 - j \\ \nu\mathbf{w}' &= m\mathbf{w} - a\mathbf{b} \\ \eta\mathbf{b}' &= v\mathbf{b} - a\mathbf{w} - (c,0)^T \end{aligned} \qquad (\Sigma^5)$$

with

- ∘ $p(v) := \pi(m/v)$, constants $a \in \mathbb{R}$, $m := \rho v > 0$,
- ∘ two parameters $j \in (-\infty, +\infty)$, $c \in [0, +\infty)$.

PROPERTIES OF THE DYNAMICAL SYSTEM (Σ^5)

(i) System (Σ^5) is gradient-like with respect to

$$P(\mathbf{u}) = \frac{1}{2}mv^2 + \frac{1}{2}m|\mathbf{w}|^2 + \frac{1}{2}v|\mathbf{b}|^2 + \int^v p(v)dv - jv - a\mathbf{b}\cdot\mathbf{w} - (c,0)^\mathsf{T}\cdot\mathbf{b},$$

i.e. (Σ^5) can be written in the form

$$B\mathbf{u}' = \nabla P(\mathbf{u})$$

with the viscosity matrix $B = \mathrm{diag}(\mu, \nu, \nu, \eta, \eta)$.

(ii) Existence of rest points:
For $c > 0$ there are $j_1^* \leq j_2^*$ such that (Σ^5) possesses

$$0 \mid 2 \mid 4 \text{ fixed points if } j \in (-\infty, j_1^*) \mid j \in (j_1^*, j_2^*) \mid j \in (j_2^*, +\infty),$$

denoted by $\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$. They are ordered according to $v_0 > v_1 > v_2 > v_3$ and $P(\mathbf{u}_0) < P(\mathbf{u}_1) < P(\mathbf{u}_2) < P(\mathbf{u}_3)$.

EXISTENCE OF VISCOUS PROFILES (for given $j \in \mathbb{R}$, $c > 0$ and $(\mu, \nu, \eta) \in (0,\infty)^3$)

Fast and Slow Shock (by Conley Index Theory)

Theorem: If the rest points $\mathbf{u}_0$ and $\mathbf{u}_1$ [resp. $\mathbf{u}_2$ and $\mathbf{u}_3$] exist, then there is a heteroclinic orbit $\mathbf{u}_0 \to \mathbf{u}_1$ [resp. $\mathbf{u}_2 \to \mathbf{u}_3$].

PROOF SKETCH.
For $k \in (P(\mathbf{u}_1), P(\mathbf{u}_2))$ the set S_{01} [resp. S_{23}] consisting of all points on complete, bounded orbits contained in $\{\mathbf{u} : P(\mathbf{u}) < k\}$ [resp. $\{\mathbf{u} : P(\mathbf{u}) > k\}$]

- contains the rest points $\mathbf{u}_{0,1}$ [resp. $\mathbf{u}_{2,3}$],
- is *isolated invariant*, and
- has Conley index $\overline{0}$ ($\equiv$ homotopy type of one-point pointed space).

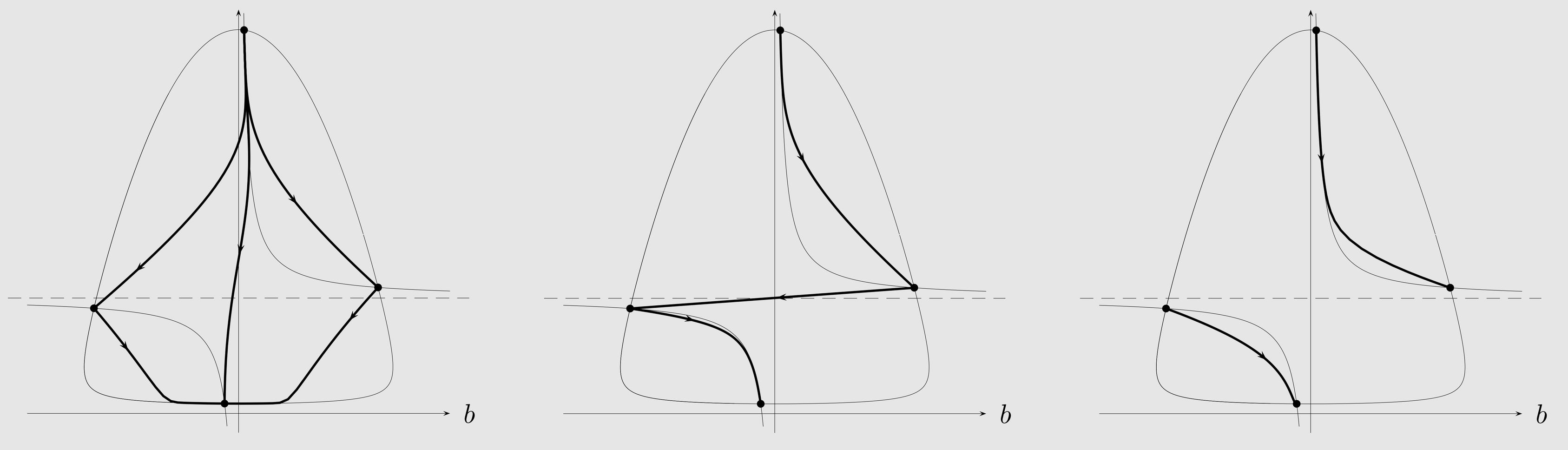
Thus, we conclude the proof by applying the following:

> ***Theorem.*** *If an isolated invariant set S of a gradient-like system contains precisely two hyperbolic rest points and has Conley index $h(S) = \overline{0}$, then there exists a heteroclinic orbit connecting these two rest points.*

Intermediate Shocks (by Geometric Singular Perturbation Theory)

Theorem: If all four rest points exist and $0 < \nu \ll \min\{\mu, \eta\}$ holds, then there is a threshold $\omega > 0$ such that:
If $\mu/\eta < \omega$, then only the int. orbits $\mathbf{u}_0 \to \mathbf{u}_2$, $\mathbf{u}_0 \to \mathbf{u}_3$, $\mathbf{u}_1 \to \mathbf{u}_3$ exist.
If $\mu/\eta = \omega$, then only the int. orbit $\mathbf{u}_1 \to \mathbf{u}_2$ exists.
If $\mu/\eta > \omega$, then no int. orbit exists.

PROOF SKETCH.
The system (Σ^5) restricted to $\{w_2 = b_2 = 0\}$ is amenable to GSPT in the sense that the existence of orbits for small $\nu > 0$ can be inferred from the existence of orbits for the reduced restricted problem ($\nu = 0$), which is indicated in the figures.



OPEN QUESTIONS

∘ What does the complete picture of the global heteroclinic bifurcation look like?

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