Binary Market Models and Discrete Wick Products

Christian Bender*
and
Robert J. Elliott†

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Abstract

We consider binary market models based on the discrete Wick product instead of the pathwise product and provide a sufficient criterion for the existence of an arbitrage. This arbitrage is explicitly constructed in the class of self-financing one-step buy-and-hold strategies, (i.e. the investor holds shares of the stock only at one time step). Then a definition of a Wick self-financing portfolio, which is the discrete analogue of the Wick-Itô integral based definition of a self-financing portfolio in the fractional Brownian motion theory, is introduced. A sufficient criterion for absence of arbitrage in the class of Wick self-financing one-step buy-and-hold strategies is proven. The results are applied to coefficients obtained from an approximation of a fractional Brownian motion with Hurst parameter $1/2 < H < 1$. Finally, we suggest that there is no economic meaning in the notion of a Wick self-financing portfolio.

Keywords: arbitrage, binary market models, discrete Wick products, Wick self-financing portfolios

1 Introduction

A fractional Brownian motion B^H with Hurst parameter $1/2 < H < 1$ inherits a long time memory, and its exponential has been considered as a stock price model in recent years. However, a fractional Brownian motion with Hurst parameter $H \neq 1/2$ is not a semimartingale and there are opportunities for an arbitrage in models in which the notion of a self-financing portfolio is defined in terms of the pathwise integral, (see Rogers, 1997; Dasgupta and Kallianpur, 2000). In contrast, using the Wick exponential of a

*Faculty of Science, Department of Mathematics and Statistics, University of Konstanz, 78457 Konstanz $\langle D126\rangle$, Germany, christian.bender@uni-konstanz.de

†Haskayne School of Business, Scurfield Hall, University of Calgary, 2500 University Drive, Calgary, Canada T2N 1N4, relliott@ucalgary.ca

fractional Brownian motion as model for the stock price and the Wick-Itô integral in the definition of a self-financing portfolio, it can be shown, that there is no arbitrage, (see Hu and Øksendal, 2000; Elliott and van der Hoek, 2001; Bender, 2002). On the other hand, as was proven in Cheridito (2001), there is an arbitrage with a linear combination of buy and hold strategies in the Wick exponential model, if one uses the usual notion of self-financing for linear combinations of buy-and-hold strategies, (which is a special case of the pathwise integral based definition).

In this paper we study binary market models of the form:

$$S_n = S_{n-1} \circ \left(1 + \mu_n + \sum_{i=1}^{n} x(n,i)\xi_i\right); \;\; S_0 = s,$$

the circle indicating either the ordinary pathwise product or the discrete Wick product. Here (ξ_i) is a family of independent binary variables with $P(\xi_i = 1) = P(\xi_i = -1)$. All coefficients are supposed to be deterministic. The scope of this paper is to derive results in the (rather simple) binary models similar to those in the continuous models based on fractional Brownian motion mentioned above.

In section 2 we fix some terminology, including the notion of a one-step buy-and-hold strategy. This is a portfolio in which shares of the stock are held at only one time step. This definition is motivated by the following intuition: Assume a discrete time market model has a long time memory. Because of this memory the stock price should heuristically go up once more, if it has increased for sufficiently many previous time steps. Hence, in this situation it should be possible to construct an arbitrage by buying one stock and selling it again after one time step.

In section 3 we recall a sufficient criterion for the existence of a self-financing one-step buy-and-hold arbitrage in the model S based on the pathwise product. Then we motivate the use of the discrete Wick product in the model S in section 4.1. A sufficient criterion for the existence of a self-financing one-step buy-and-hold arbitrage in a sequence of models S based on the discrete Wick product is established in section 4.2.

In section 5 we introduce a notion of a Wick self-financing portfolio, which is the discrete time analogue of the Wick-Itô integral definition of a self-financing portfolio in fractional Brownian motion theory. A sufficient criterion for the absence of arbitrage in the class of Wick self-financing, one-step buy-and-hold strategies is provided for the model S based on the discrete Wick product.

The results are applied to a sequence of coefficients obtained from an approximation of a fractional Brownian motion with Hurst parameter $1/2 < H < 1$, derived by Sottinen (2001), in section 6. Theorem 6.2 can be regarded as the discrete analogue of the results mentioned in the continuous fractional Brownian motion models. Roughly speaking, the theorem implies that an arbitrage free model cannot be obtained when using self-financing portfolios, but it can when using Wick self-financing portfolios. However, as we try to argue in section 7, the definition of a Wick self-financing portfolio appears artificial and seems to have no economic meaning.

2 Preliminaries

Define, for $N \in \mathbb{N}$, $\Omega_N = \{-1,1\}^N$, let $\mathcal{A}_N$ be the power set of Ω_N and P_N be the uniform probability measure on $\mathcal{A}_N$. Then

$$\xi_i(\omega) = \omega_i, \;\; 1 \leq i \leq N,$$

is a family of independent binary variables with $P_N(\xi_i = 1) = P_N(\xi_i = -1)$. By Holden et al. (1993) every random variable $Y \in L^2(\Omega_N, P_N)$ has a unique expansion of the form

$$Y = \sum_{A \subset \{1,\ldots,N\}} \left(Y(A) \prod_{i \in A} \xi_i \right); \;\; Y(A) \in \mathbb{R}, \tag{1}$$

the so called *Walsh decomposition*.

A *discounted* (B,S) *market* on $(\Omega_N, \mathcal{A}_N, P_N)$ is a pair of $L^2(\Omega_N, P_N)$-valued random vectors (B,S) such that $B_n = 1$ and S_n is $\mathcal{F}_n$-measurable for all $0 \leq n \leq N$. Here $\mathcal{F}_n$ denotes the filtration generated by $(\xi_1, \ldots, \xi_n)$. B is a riskless asset, the *bond*, whereas the risky asset S is called the *stock*. A *portfolio* is a pair of $L^2(\Omega_N, P_N)$-valued random vectors $\pi = (F,G)$ such that F_n and G_n are $\mathcal{F}_{n-1}$-measurable for all $0 \leq n \leq N$. F_n and G_n are the numbers of bonds, resp. stocks, that an investor holds at time n. Hence, the *value* of a portfolio π at time $0 \leq n \leq N$ is given by:

$$V_n^\pi = F_n + G_n S_n. \tag{2}$$

Definition 2.1. A portfolio $\pi = (F,G)$ is called *self-financing*, if the corresponding value process satisfies:

$$V_n^\pi = V_{n-1}^\pi + G_n(S_n - S_{n-1}); \;\; 1 \leq n \leq N. \tag{3}$$

This means, that the only change in the wealth comes from the change in the stock, (as the change in the bond is 0 by definition).

In this paper we investigate, whether or not an investor has an opportunity for a riskless gain in the following sense:

Definition 2.2. A portfolio $\pi = (F,G)$ is said to be an *arbitrage* if there is a time step n, $1 \leq n \leq N$, such that the corresponding value process satisfies: $V_0^\pi = 0$, $V_n^\pi(\omega) \geq 0$ for all $\omega \in \Omega$ and $V_n^\pi(\eta) > 0$ for at least one $\eta \in \Omega$.

We are going to consider models, in which the stock price S has a long time memory. The intuition, why there should be an arbitrage in models with a long time memory, is as follows: If the stock price goes up long enough, it will go up once more due to the long time memory. For this reason we consider one-step buy-and-hold strategies in this paper only:

Definition 2.3. We call a portfolio $\pi = (F, G)$ *one-step buy-and-hold strategy*, if there is a time step n, $0 \leq n \leq N$, such that $G_k = 0$ for all $k \neq n$.

3 Binary Market Models with Memory Based on Pathwise Products

Consider first the following usual binary market model based on the pathwise product: We assume the stock is given by:

$$S_n = S_{n-1}(1 + \mu_n + X_n); \quad S_0 = s. \tag{4}$$

Here the *initial price* s is a positive real, the function $\mu\colon \{1, \ldots, N\} \to \mathbb{R}$ is usually interpreted as the *drift* of the stock, and the *volatility* is thought to be given by X. In the following we assume X to be of the form:

$$X_n = \sum_{i=1}^{n} x(n, i)\xi_i \tag{5}$$

with real numbers $x(n, i)$, $1 \leq i \leq n \leq N$. Moreover, we assume that $X_n + \mu_n > -1$ for all $1 \leq n \leq N$ to ensure that the stock price stays positive.

In this situation we have the following sufficient criterion for an arbitrage:

Theorem 3.1. *Assume that there is a time step* $1 \leq n \leq N$ *such that*

$$\sum_{i=1}^{n-1} |x(n, i)| > |x(n, n)|. \tag{6}$$

Then there is an arbitrage at time step n *in the class of self-financing one-step buy-and-hold-strategies in the model given by (4)–(5).*

The proof is similar to the argument in Sottinen (2001, theorem 5). We include it for the reader's convenience.

Proof. Assume first, that $\mu_n \geq 0$. Let

$$A_+ = \{\text{sign}(x(n, 1))\} \times \cdots \{\text{sign}(x(n, n-1))\} \times \{-1, 1\} \times \cdots \times \{-1, 1\},$$

(using the left continuous version of the sign function), and let $\omega \in A_+$. Then by (6)

$$X_n(\omega) + \mu_n > 0$$

and hence, $S_n(\omega) > S_{n-1}(\omega)$. A self-financing one-step buy-and-hold arbitrage can now be constructed as follows: If $\omega \in A_+$ borrow the amount $S_{n-1}(\omega)$ from the bank and buy one stock at time step n. Then sell the stock at time step $n+1$. If $\omega \in A_+^C$ do nothing.

In the case $\mu_n < 0$ consider the set

$$A_- = \{-\operatorname{sign}(x(n,1))\} \times \cdots \{-\operatorname{sign}(x(n,n-1))\} \times \{-1,1\} \times \cdots \times \{-1,1\}$$

and proceed analogously by short selling a stock at time step n, if $\omega \in A_-$. □

4 Binary Market Models with Memory Based on Discrete Wick Products

4.1 Motivation

We shall now motivate, why the model (4)-(5) should be modified. To this end let us calculate $E[S_2]$:

$$\begin{aligned} E[S_2] &= E\left[S_1\left(1+\mu_2+\sum_{i=1}^{2}x(2,i)\xi_i\right)\right] \\ &= sE\left[(1+\mu_1+x(1,1)\xi_1)\left(1+\mu_2+\sum_{i=1}^{2}x(2,i)\xi_i\right)\right] \\ &= s(1+\mu_1)(1+\mu_2)+sx(1,1)x(2,1), \end{aligned}$$

because $\xi_1^2 = 1$. This shows, that not only μ, but also X contributes to the drift, contrary to the usual interpretation, that μ alone determines the drift (e.g. Sottinen, 2001, p. 351). More general, the fact that $\xi_i^2 = 1$ has the effect that over time more and more randomness fades out and contributes to the drift. One might suspect that this is the reason for the existence of an arbitrage.

The easiest way to get rid of this effect is to introduce the discrete Wick product:

Definition 4.1. Let $A, B \subset \{1, \dots, N\}$. Then the *discrete Wick product* is defined as

$$\prod_{i\in A}\xi_i \diamond \prod_{i\in B}\xi_i = \begin{cases} \prod_{i\in A\cup B}\xi_i, & \text{if } A\cap B = \emptyset \\ 0, & \text{otherwise.} \end{cases}$$

The discrete Wick product is then extended to $X, Y \in L^2(\Omega_N, P_N)$ by linearity using their Walsh decomposition (1). E.g. for constants a, b, c:

$$(a\xi_1\xi_2 + b\xi_2) \diamond c\xi_1 = ac\left[(\xi_1\xi_2)\diamond\xi_1\right] + bc[\xi_2\diamond\xi_1] = bc\xi_2\xi_1$$

Note, that in particular for a constant a:

$$a \diamond \prod_{i \in B} \xi_i = \left(a \prod_{i \in \emptyset} \xi_i \right) \diamond \prod_{i \in B} \xi_i = a \left(\prod_{i \in \emptyset} \xi_i \diamond \prod_{i \in B} \xi_i \right) = a \cdot \prod_{i \in B} \xi_i.$$

We should also mention the obvious fact, that for $Y \in L^2(\Omega_N, P_N)$ the expectation is given in terms of its Walsh decomposition (1) by

$$E[Y] = Y(\emptyset).$$

A simple consequence is, that for $X, Y \in L^2(\Omega_N, P_N)$:

$$E[X \diamond Y] = E[X] \cdot E[Y]$$

For more details on the discrete Wick product the reader is referred to Holden et al. (1993).

We can now modify the model in (4)–(5) by replacing (4) by

$$S_n = S_{n-1} \diamond (1 + \mu_n + X_n); \quad S_0 = s. \tag{7}$$

As $X, Y > 0$ does not imply $X \diamond Y > 0$, we shall drop the assumption $\mu_n + X_n > -1$ and allow the stock price to be negative for the moment.

Remark 4.1. Suppose $X_n = \sigma_n (B_n^{H,N} - B_{n-1}^{H,N})$ with deterministic σ and $B^{H,N}$ an approximation of a fractional Brownian motion B^H with Hurst parameter $1/2 < H < 1$. Then (7) is the discrete analogue of the fractional Doléans-Dade equation, (the stochastic integral in the fractional Wick-Itô sense):

$$S_t = s + \int_0^t \sigma(s) S_s dB_s^H + \int_0^t \mu(s) S_s ds.$$

For appropriate coefficients its solution is given by

$$S_t := s \exp \left\{ \int_0^t \mu(s) ds - \frac{1}{2} \left| M_-^H (\mathbf{1}(0,t)\sigma) \right|^2_{L^2(\mathbb{R})} + \int_0^t \sigma(s) dB_s^H \right\},$$

where M_-^H is, (up to a constant), a fractional integral operator of order $H - 1/2$, see Bender (2002, corollary 5.6). Exponentials of this kind have been considered as models for stock prices in Hu and Øksendal (2000), Elliott and van der Hoek (2001) and Bender (2002).

It is easy to verify that S_n given by (7) allows a decomposition of the form (1). To be more precise,

$$S_n = \sum_{k=0}^{n} \sum_{\{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}} S_n(\{i_1, \ldots, i_k\}) \xi_{i_1} \cdots \xi_{i_k}. \tag{8}$$

Throughout this paper we do not need to calculate the coefficients of this decomposition explicitly. But it is straightforward, that

$$E\left[S_n\right] = S_n(\emptyset) = s_0 \prod_{i=1}^{n} (1 + \mu_n). \tag{9}$$

Hence, the drift of the stock S_n is now given by μ only.

4.2 The Main Result on Arbitrage

We are now going to prove the existence of an arbitrage in a sequence of models S^N given by (7) and (5) under appropriate conditions on the coefficients:

Theorem 4.2. *Let $N \in \mathbb{N}$, $H \in (0,1)$. Define a sequence of market models S^N_n, $n \leq N$ by (7) and (5) and assume the coefficients μ^N_n and $x_N(n,i)$ of S^N satisfy the following conditions:*
(i) $\mu^N : \{1, \ldots, N\} \to \mathbb{R}$ fulfills:

$$|\mu^N_n| \leq \frac{K}{N}; \quad 1 \leq n \leq N \tag{10}$$

for a constant K independent of N.
(ii) There are $y_u(n,i)$ and $y_l(n,i)$ independent of N, such that for all $N \in \mathbb{N}$:

$$N^{-H} y_l(n,i) \leq |x_N(n,i)| \leq N^{-H} y_u(n,i); \quad 1 \leq i \leq n \leq N \tag{11}$$

and

$$\sum_{i=1}^{n-1} y_l(n,i) > y_u(n,n) \tag{12}$$

for some $n \in \mathbb{N}$.

Then there exists an arbitrage in the class of self-financing one-step buy-and-hold strategies at time step n in the model S^N for sufficiently large N.

Proof. Choose an $n \in \mathbb{N}$, that satisfies (12), let $N \geq n$ and consider the decomposition (8) of S^N_{n-1}, i.e.

$$S^N_{n-1} = \sum_{k=0}^{n-1} \sum_{\{i_1,\ldots,i_k\} \subset \{1,\ldots,n-1\}} S^N_{n-1}(\{i_1,\ldots,i_k\}) \xi_{i_1} \cdots \xi_{i_k}.$$

From (10) and (11) we may conclude, that for $k > 0$:

$$S^N_{n-1}(\{i_1,\ldots,i_k\}) = O(N^{-H}).$$

Furthermore by (9) and (10):

$$S^N_{n-1}(\emptyset) = s_0 \prod_{i=1}^{n} (1 + \mu_n) = s_0 + O(N^{-1}).$$

Hence,

$$S^N_{n-1} = s_0 + O(N^{-H}).$$

This implies:

$$\begin{aligned} S_n^N - S_{n-1}^N &= \mu_n S_{n-1}^N + X_n^N \diamond S_{n-1}^N \\ &= s_0\mu_n + \mu_n O(N^{-H}) + s_0 \sum_{i=1}^{n} x_N(n,i)\xi_i \\ &\quad +(S_{n-1} - s_0) \diamond \sum_{i=1}^{n} x_N(n,i)\xi_i \\ &= s_0 \sum_{i=1}^{n} x_N(n,i)\xi_i + O(N^{-(1\wedge 2H)}), \end{aligned} \tag{13}$$

since for $k > 1$

$$\begin{aligned} & [S_{n-1}^N(\{i_1,\dots,i_k\})\xi_{i_1}\cdots\xi_{i_k}] \diamond [x_N(n,i)\xi_i] \\ = & \begin{cases} x_N(n,i)S_{n-1}^N(\{i_1,\dots,i_k\})\xi_{i_1}\cdots\xi_{i_k}\xi_i, & \text{if } i \neq i_k \text{ for all } k \\ 0, & \text{otherwise} \end{cases} \\ = & \ O(N^{-2H}) \end{aligned}$$

and

$$(S_{n-1}^N(\emptyset) - s_0) \diamond (x_N(n,i)\xi_i) = (S_{n-1}^N(\emptyset) - s_0) \cdot x_N(n,i)\xi_i = O(N^{-1-H}).$$

Let

$$\omega \in \{\mathrm{sign}(x(n,1))\} \times \cdots \{\mathrm{sign}(x(n,n-1))\} \times \{-1,1\} \times \cdots \times \{-1,1\}.$$

Then we deduce from (13) and (12)

$$\begin{aligned} & (S_n^N - S_{n-1}^N)(\omega) \\ = \ & s_0 \sum_{i=1}^{n-1} |x_N(n,i)| + x_N(n,n)\xi_n(\omega) + O(N^{-(1\wedge 2H)}) \\ \geq \ & N^{-H}\left[s_0 \sum_{i=1}^{n-1} y_l(n,i) - s_0 y_u(n,n) + O(N^{-[(1-H)\wedge H]}) \right] \\ > \ & 0, \end{aligned}$$

if N is sufficiently large, since $0 < H < 1$. Thus, a self-financing one-step buy-and-hold arbitrage at time step n can be constructed as in theorem 3.1 in the model S^N, if N is sufficiently large. □

5 Discrete Wick Products and Self-financing Portfolios

In the previous section we used the ordinary definition of a self-financing portfolio (definition 2.1). Substituting (7) in (3) and summing we get:

$$V_n^\pi = V_0^\pi + \sum_{k=1}^{n} G_k S_{k-1}\mu_k + \sum_{k=1}^{n} G_k(S_{k-1} \diamond X_k). \tag{14}$$

Note that the second sum in (14) cannot be interpreted as a discrete version of the fractional Wick-Itô integral. However, if X_k is chosen as an approximation of the increments of a fractional Brownian motion, the discrete version of the Wick-Itô integral would be:

$$\sum_{k=1}^{n}(G_k S_{k-1}) \diamond X_k \tag{15}$$

Recalling the Wick-Itô integral based definitions of a self-financing portfolio in Hu and Øksendal (2000), Elliott and van der Hoek (2001) and Bender (2002) this motivates the following definition:

Definition 5.1. A portfolio $\pi = (F, G)$ is called *Wick self-financing*, if the corresponding value process V_n^π fulfills for all n, $1 \le n \le N$

$$V_n^\pi = V_{n-1}^\pi + G_n S_{n-1}\mu_n + (G_n S_{n-1}) \diamond X_n. \tag{16}$$

We can now prove a sufficient criterion for absence of arbitrage in the class of one-step buy-and-hold strategies using the notion of Wick self-financing portfolios:

Theorem 5.2. *Consider the stock price model S given by (7) and (5), and assume that for all $1 \le n \le N$ $|x(n,n)| > |\mu_n|$. Then there is no arbitrage in the class of Wick self-financing one-step buy-and-hold strategies.*

Proof. Let $\pi = (F, G)$ be a Wick self-financing one-step buy-and-hold strategy for time k with $V_0^\pi = 0$. By (16) $V_i^\pi = 0$ for all $i < k$ and $V_i^\pi = V_k^\pi$ for all $i \ge k$. Hence, it is sufficient to prove that π is not an arbitrage at time step k. But since $V_{k-1}^\pi = 0$ and π is Wick self-financing we obtain

$$\begin{aligned} V_k^\pi &= G_k S_{k-1}\mu_k + (G_k S_{k-1}) \diamond X_k \\ &= G_k S_{k-1}\mu_k + \sum_{i=1}^{k} x(k,i)(G_k S_{k-1}) \diamond \xi_i \\ &= G_k S_{k-1}\left[\mu_k + x(k,k)\xi_k\right] \\ &\quad + \sum_{i=1}^{k-1} x(k,i)(G_k S_{k-1}) \diamond \xi_i. \end{aligned} \tag{17}$$

Note, that the last identity holds, since $G_k S_{k-1}$ is $\mathcal{F}_{k-1}$-measurable and hence $G_k S_{k-1} \diamond \xi_k = G_k S_{k-1} \xi_k$.

Suppose first that there is an $\eta \in \Omega$ such that $V_k^\pi(\eta) > 0$ and

$$\left(\sum_{i=1}^{k-1} x(k,i)(G_k S_{k-1}) \diamond \xi_i \right)(\eta) \leq 0.$$

Let η' be the path that equals η but with η_k replaced by $-\eta_k$. Then it follows from the assumption on $x(k,k)$ that

$$G_k S_{k-1} \left[\mu_k + x(k,k)\xi_k \right](\eta') < 0,$$

since

$$G_k S_{k-1} \left[\mu_k + x(k,k)\xi_k \right](\eta) > 0.$$

Consequently, $V_k^\pi(\eta') < 0$.

Suppose now, that there is an $\eta \in \Omega$ such that $V_k^\pi(\eta) > 0$ and

$$\left(\sum_{i=1}^{k-1} x(k,i)(G_k S_{k-1}) \diamond \xi_i \right)(\eta) > 0.$$

As

$$E\left[\sum_{i=1}^{k-1} x(k,i)(G_k S_{k-1}) \diamond \xi_i \right] = 0,$$

(recall that $E[X \diamond Y] = EX \cdot EY$), there is an $\omega \in \Omega$ such that

$$\left(\sum_{i=1}^{k-1} x(k,i)(G_k S_{k-1}) \diamond \xi_i \right)(\omega) < 0.$$

Now $V_k^\pi(\omega) < 0$ or $V_k^\pi(\omega') < 0$.

Hence, whenever there is a path η with $V_k^\pi(\eta) > 0$, there is a path $\tilde{\eta}$ such that $V_k^\pi(\tilde{\eta}) < 0$. Consequently, π is not an arbitrage at time step k. $\square$

Remark 5.1. Note, that there is obviously no Wick self-financing arbitrage at all in the model (7), (5), if the drift μ equals zero at all times.

6 An Example Based on Fractional Brownian Motion

Fix $1/2 < H < 1$ and define for $N \in \mathbb{N}$, $1 \leq n \leq N$:

$$b^{H,N}(n,i) = C_H(H-1/2)N \int_{\frac{i-1}{N}}^{\frac{i}{N}} s^{-(H-1/2)} \int_s^{\frac{n}{N}} u^{H-1/2}(u-s)^{H-3/2} du ds, \tag{18}$$

if $i \le n$, and $b^{H,N}(n,i) = 0$, if $i > n$, where the constant C_H is given by

$$C_H = \left(\frac{2H\Gamma(3/2-H)}{\Gamma(H+1/2)\Gamma(2-2H)} \right)^{1/2}.$$

Using a representation for fractional Brownian motion B^H obtained in Norros et al. (1999) it is proven in Sottinen (2001), that the sequence of processes

$$B_t^{H,N} = \sum_{i=1}^{n} b^{H,N}(n,i) N^{-1/2} y_i; \quad \frac{n-1}{N} \le t < \frac{n}{N},\ 1 \le n \le N,$$

(where (y_i) is a family of independent identically distributed variables with zero mean and variance 1), weakly converges to a fractional Brownian motion B_t^H, $t \in [0,1]$, in the Skorohod space.

This motivates our choice:

$$X_n^N = \sum_{i=1}^{n} x_N(n,i)\xi_i = \sum_{i=1}^{n} \sigma(b^{H,N}(n,i) - b^{H,N}(n-1,i)) N^{-1/2} \xi_i \tag{19}$$

for some positive constant σ that determines the intensity of the volatility.

Based on X^N we shall consider the sequences of markets:

$$S_n^N = S_{n-1}^N (1 + \frac{\mu}{N} + X_n^N); \quad S_0^N = s. \tag{20}$$

and

$$\tilde{S}_n^N = \tilde{S}_{n-1}^N \diamond (1 + \frac{\mu}{N} + X_n^N); \quad \tilde{S}_0^N = s. \tag{21}$$

We assume s and, for sake of simplicity, also μ to be positive constants.

We need the following estimates:

Lemma 6.1. *Define for* $1 \le i \le n \in \mathbb{N}$:

$$y_l(n,i) = \begin{cases} C_H\sigma \left[(n+1-i)^{H-1/2} - (n-i)^{H-1/2} \right], & i < n \\ \frac{C_H\sigma}{H+1/2}, & i = n \end{cases}$$

and

$$y_u(n,i) = \begin{cases} \frac{C_H\sigma}{H+1/2} \left(\frac{n}{i-1} \right)^{H-1/2} \left[(n+1-i)^{H+1/2} - (n-i)^{H+1/2} \right], & i > 1 \\ \frac{C_H\sigma}{3/2-H} n^{2H-1}, & i = 1. \end{cases}$$

Then for all $1 \le i \le n \le N$:

$$N^{-H} y_l(n,i) \le x_N(n,i) \le N^{-H} y_u(n,i).$$

Proof. Recall, that

$$\begin{aligned}
& x_N(n,i) \\
= \quad & C_H(H-1/2)\sigma N^{1/2}\int_{\frac{i-1}{N}}^{\frac{i}{N}} s^{-(H-1/2)}\int_{s\vee\frac{n-1}{N}}^{\frac{n}{N}} u^{H-1/2}(u-s)^{H-3/2}duds.
\end{aligned}$$

We first prove the lower bound for $i=n$:

$$\begin{aligned}
& \int_{\frac{n-1}{N}}^{\frac{n}{N}} s^{-(H-1/2)}\int_{s}^{\frac{n}{N}} u^{H-1/2}(u-s)^{H-3/2}duds \\
\geq \quad & \int_{\frac{n-1}{N}}^{\frac{n}{N}}\int_{s}^{\frac{n}{N}}(u-s)^{H-3/2}duds \\
= \quad & (H-1/2)^{-1}(H+1/2)^{-1}N^{-H-1/2}
\end{aligned}$$

. The lower bound for $i<n$ directly follows from Sottinen (2001, formula (9), p. 353).

The upper bound for $i=1$ can be proven as follows:

$$\begin{aligned}
& \int_{0}^{\frac{1}{N}} s^{-(H-1/2)}\int_{s\vee\frac{n-1}{N}}^{\frac{n}{N}} u^{H-1/2}(u-s)^{H-3/2}duds \\
\leq \quad & (H-1/2)^{-1}\left(\frac{n}{N}\right)^{H-1/2}\int_{0}^{\frac{1}{N}} s^{-(H-1/2)}\left(\frac{n}{N}-s\right)^{H-1/2}ds \\
\leq \quad & (H-1/2)^{-1}\left(\frac{n}{N}\right)^{2H-1}\int_{0}^{\frac{1}{N}} s^{-(H-1/2)}ds \\
= \quad & (3/2-H)^{-1}(H-1/2)^{-1}n^{2H-1}N^{-H-1/2}.
\end{aligned}$$

It remains to prove the upper bound for $i>1$:

$$\begin{aligned}
& \int_{\frac{i-1}{N}}^{\frac{i}{N}} s^{-(H-1/2)}\int_{s\vee\frac{n-1}{N}}^{\frac{n}{N}} u^{H-1/2}(u-s)^{H-3/2}duds \\
\leq \quad & \left(\frac{N}{i-1}\right)^{H-1/2}\left(\frac{n}{N}\right)^{H-1/2}\int_{\frac{i-1}{N}}^{\frac{i}{N}}\int_{s\vee\frac{n-1}{N}}^{\frac{n}{N}}(u-s)^{H-3/2}duds \\
\leq \quad & (H-1/2)^{-1}\left(\frac{n}{i-1}\right)^{H-1/2}\int_{\frac{i-1}{N}}^{\frac{i}{N}}\left(\frac{n}{N}-s\right)^{H-1/2}ds \\
= \quad & (H+1/2)^{-1}(H-1/2)^{-1}\left(\frac{n}{i-1}\right)^{H-1/2}N^{-H-1/2} \\
& \times\left[(n+1-i)^{H+1/2}-(n-i)^{H+1/2}\right].
\end{aligned}$$

□

We can now prove the following theorem, which can be regarded as the discrete time analogue of the continuous time results in markets based on (Wick-)geometric fractional Brownian motion:

Theorem 6.2. *Let $N \in \mathbb{N}$ be sufficiently large. Then:*
(i) There is an arbitrage in the class of self-financing one-step buy-and-hold strategies in the model S^N.
(ii) There is an arbitrage in the class of self-financing one-step buy-and-hold strategies in the model $\tilde{S}^N$.
(iii) There is no arbitrage in the class of Wick self-financing one-step buy-and-hold strategies in the model $\tilde{S}^N$.

Remark 6.1. Note, that (i) has been proven in Sottinen (2001, theorem 5).

Proof. Using the bounds from the previous lemma, we obtain, (since the sum telescopes):

$$\sum_{i=1}^{n-1} y_l(n,i) - y_u(n,n) = C_H\sigma \left[n^{H-1/2} - 1 - \left(H + \frac{1}{2}\right)^{-1} \left(\frac{n}{n-1}\right)^{H-1/2} \right]$$

Thus, $\sum_{i=1}^{n-1} y_l(n,i) - y_u(n,n)$ converges to $+\infty$. Consequently,

$$\sum_{i=1}^{n-1} y_l(n,i) - y_u(n,n) > 0 \tag{22}$$

for sufficiently large $n \in \mathbb{N}$. In combination with the previous lemma, we see that theorem 4.2 applies. Hence, (ii) follows.

In view of (22) and the bounds from the previous lemma it is also obvious that (6) is satisfied, if N is sufficiently large. This implies (i).

As $H < 1$ we obtain from the above lemma,

$$|x_N(n,n)| \geq N^{-H} \frac{C_H\sigma}{H + 1/2} > \frac{\mu}{N}$$

for sufficiently large N. Thus, theorem 5.2 implies (iii). □

7 Comparison between Self-financing and Wick Self-financing Portfolios

From theorem 6.2 we see one has to use the notion of a Wick self-financing portfolio in order to obtain an arbitrage free model. While the interpretation of a self-financing portfolio is straightforward, however, an economic meaning of a Wick self-financing portfolio seems hard to find. We shall now illustrate this problem:

Let us consider the model (7), (5). Moreover, assume there is a pair $\Pi = (v, G)$, where $v \in \mathbb{R}$. As before, G is a $L^2(\Omega_N, P_N)$-valued random vector, such that G_n is $\mathcal{F}_{n-1}$ measurable for every $0 \leq n \leq N$. We interpret v as the initial wealth of an investor and G_n as the number of stocks held by him at time n. Obviously, Π defines a unique self-financing portfolio $\pi = (F, G)$ with initial wealth $V_0^\pi = v$, and also a unique Wick self-financing portfolio $\tilde{\pi} = (\tilde{F}, G)$ with initial wealth $V_0^{\tilde{\pi}} = v$. As the notion of a self-financing portfolio is natural, we refer to the random vector V^π as the *real value* of Π. The random vector $V^{\tilde{\pi}}$ is called the *Wick value* of Π. Hence, the problem of explaining the notion of Wick self-financing can be reformulated as the problem of giving the difference $V^\pi - V^{\tilde{\pi}}$ an economic meaning.

Let us assume that there is no drift in the stock price, i.e. $\mu_n = 0$ for all $0 \leq n \leq N$ and that $S_0 = 1$. Moreover let $x(n, n) > 0$ for all $0 \leq n \leq N$ corresponding to the intuition that $\xi_n = 1$ is the up-state of S_n. We consider the pair $\Pi = (v, G)$ with $v = 0$, $G_2 = \xi_1$ and $G_n = 0$ for $n \neq 2$. A straightforward calculation yields:

$$V_2^\pi = x(2,1) + x(1,1)x(2,2)\xi_2 + x(2,2)\xi_1\xi_2$$

and

$$V_2^\pi - V_2^{\tilde{\pi}} = x(2,1) - x(1,1)x(2,1)\xi_1.$$

It seems unlikely that $V_2^\pi - V_2^{\tilde{\pi}}$ can be interpreted as a transaction cost, because the sign of $V_2^\pi - V_2^{\tilde{\pi}}$ depends on the sign of $x(2,1)$. Negative transaction costs are not plausible. Another possible interpretation is to introduce costly information. Here the idea might be the better the investor uses the information of the past, the more he has to pay for it. Following this idea the term $x(2,1)$ in $V_2^\pi - V_2^{\tilde{\pi}}$ is a 'good' term, because it is the expectation of the real value of Π at time step 2. However, the term $x(1,1)x(2,1)\xi_1$ is rather strange, since no random term involving $x(2,1)$ is part of the real value of Π at time step 2. The sign of $x(2,1)$ would determine, whether the investor does better, if $\xi_1 = 1$ or $\xi_1 = -1$. It seems, that this cannot be justified, either.

In conclusion, it appears the notion of a Wick self-financing portfolio is an artificial, mathematical construct with no economic meaning. Although we discussed the meaning of a Wick self-financing portfolio only in this simple, discrete model, our results should at least increase the scepticism that there could be a good interpretation of the continuous time definition of a Wick self-financing portfolio.

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