Fachbereich Mathematik und Statistik
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WS 2018 / 2019

Universität Konstanz

Real Algebraic Geometry I

Exercise Sheet 6
Semialgebraic sets

Exercise 21
(4 points)
Let (K,P) be an ordered field and let R be the real closure of (K,P). Recall that R can be equipped with the order topology and K can be considered as a topological subspace of R.

(a) Suppose that (K,P) is Archimedean. Show that K is dense in R, i.e. that R is the topological closure of K in R.

(b) Construct an ordered field which is not dense in its real closure.

(Hint: Take a suitable field Q and consider $Q(\mathrm{x})$ with a suitable ordering.)

Exercise 22
(4 points)
Let R be a real closed field and let $S(\underline{T},\underline{X})$ be the system

$$T_2X_1^2 + T_1X_2^2 + T_1T_2X_1 + 1 = 0,$$

where $\underline{T} = (T_1,T_2)$ and $\underline{X} = (X_1,X_2)$. Find systems of equalities and inequalities $S_1(\underline{T}),\ldots,S_\ell(\underline{T})$ with coefficients in $\mathbb{Q}$ such that

$$\forall \underline{T} \in R^2 : \left[\left(\exists \underline{X} \in R^2 : S\left(\underline{T},\underline{X}\right)\right) \Longleftrightarrow \bigvee_{i=1}^{\ell} S_i\left(\underline{T}\right)\right].$$

Exercise 23
(4 points)
Let R be a real closed field.

(a) Show that the semialgebraic sets in R are exactly the finite unions of points in R and open intervals with endpoints in $R \cup \{\infty, -\infty\}$.

(b) Let $m \in \mathbb{N}$ and let A be a semialgebraic subset of R^m. Show that for some $n \in \mathbb{N}$, there is an algebraic set $B \subseteq R^{m+n}$ such that $\pi(B) = A$, where $\pi : R^{m+n} \to R^m$ is the projection map introduced in Lecture 11.

(Hint: Find a polynomial $f \in R[\underline{t},\underline{x}]$, where $\underline{t} = (t_1,\ldots,t_m)$ and $\underline{x} = (x_1,\ldots,x_n)$, such that $A = \{\underline{t} \in R^m \mid \exists \underline{x} \in R^n : f(\underline{t},\underline{x}) = 0\}$.)

Exercise 24
(4 points)
Let $\exp : \mathbb{R} \to \mathbb{R}$, $x \mapsto \mathrm{e}^x$ be the standard exponential function on $\mathbb{R}$.

(a) Let $S \subseteq \mathbb{R}$ be infinite and $g \in \mathbb{R}[X, Y]$ such that for any $x \in S$

$$g(x, \exp(x)) = 0.$$

Show that $g = 0$.

(Hint: If S is bounded, use the identity theorem of complex analysis: Let f and g be holomorphic, i.e. differentiable, functions from $\mathbb{C}$ to $\mathbb{C}$. Suppose that the set $\{x \in \mathbb{C} \mid f(x) = g(x)\}$ has a limit point in $\mathbb{C}$. Then f and g coincide on $\mathbb{C}$, i.e. $f(x) = g(x)$ for any $x \in \mathbb{C}$.)

(b) Deduce that exp is not a semialgebraic map.

Please hand in your solutions by ***Thursday, 06 December 2018, 08:15h*** *(postbox 16 in F4).*