Fachbereich Mathematik und Statistik
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Universität Konstanz

Real Algebraic Geometry I

Exercise Sheet 9
Real closed fields II

Exercise 33
(4 points)
Let R be a real closed field and let $S(\underline{T}, \underline{X})$ be the system

$$T_2X_1^2 + T_1X_2^2 + T_1T_2X_1 - 1 = 0,$$

where $\underline{T} = (T_1, T_2)$ and $\underline{X} = (X_1, X_2)$. Find systems of equalities and inequalities $S_1(\underline{T}), \ldots, S_\ell(\underline{T})$ with coefficients in $\mathbb{Q}$ such that

$$\forall \underline{T} \in R^2: \left[\left(\exists \underline{X} \in R^2 : S\left(\underline{T}, \underline{X}\right) \right) \Longleftrightarrow \bigvee_{i=1}^{\ell} S_i\left(\underline{T}\right) \right].$$

Exercise 34
(4 points)
An ordered field $(K, \leq)$ is called **Euclidean** if any non-negative element has a square root in K, i.e. for any $x \in K$ with $x \geq 0$ there is some $y \in K$ such that $y^2 = x$. Construct a Euclidean ordered field which is *not* real closed.

Exercise 35
(4 points)

(a) Let T be the set of all elements in $\mathbb{R}$ which are transcendental over $\mathbb{Q}$. Show that there is a bijection between T and $\mathbb{R}$, i.e. that T and $\mathbb{R}$ have the same cardinality.

(b) Show that for any set A, there is a set P_A with greater cardinality than A, i.e. there is no surjection from A to P_A. Deduce that there are at least countably infinitely many distinct uncountable cardinalities.

Please hand in your solutions by ***Thursday, 10 January 2019, 08:15h*** *(postbox 16 in F4).*