Fachbereich Mathematik und Statistik
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Universität Konstanz

Real Algebraic Geometry I

Exercise Sheet 12
PSD- and SOS polynomials II

Exercise 43
(4 points)
The aim of this exercise is to prove the Spectral Theorem for real closed fields. Let R be a real closed field. Let $n \in \mathbb{N}$ and $M_n(R)$ the set of all $(n \times n)$-matrices with coefficients in R. Show that for every symmetric matrix $A \in M_n(R)$, there is a matrix $S \in M_n(R)$ and a diagonal matrix $D \in M_n(R)$ such that

$$S^T S = I \text{ and } A = SDS^T.$$

Exercise 44
(4 points)

(a) Show that $f(x,y) = x^4y^2 + x^2y^4 - 3x^2y^2 + 1 \in \mathbb{R}[x,y]$ is not sos.
(Hint: Assume, for a contradiction, that f is sos and compare coefficients. Note that $f(x,0) = f(0,y) = 1$.)

(b) Deduce that the Motzkin form $M(x,y,z) = z^6 + x^4y^2 + x^2y^4 - 3x^2y^2z^2 \in \mathbb{R}[x,y,z]$ is not sos.

Exercise 45
(4 points)
Show that for all $n \in \mathbb{N}$ and for all $\alpha_1, \ldots, \alpha_n, x_1, \ldots, x_n \in \mathbb{R}^{\geq 0} = [0, \infty[$,

$$\sum_{i=1}^{n} \alpha_i = 1 \implies \sum_{i=1}^{n} \alpha_i x_i - \prod_{i=1}^{n} x_i^{\alpha_i} \geq 0.$$

Exercise 46
(4 points)
Show that the symmetric quaternary quartic $F \in \mathbb{R}[\underline{x}]$ given by

$$F(x_1, x_2, x_3, x_4) = \sum_{j=2}^{4} \sum_{i<j} x_i^2 x_j^2 + \sum_{k=2}^{4} \sum_{j<k} \sum_{k \neq i \neq j} x_i^2 x_j x_k - 2x_1 x_2 x_3 x_4$$

is psd but not sos.
(Hint: Recall Proposition 4.2 of Lecture 23.)

Bonus Exercise
(3 points)
Let R be a real closed field, $n \in \mathbb{N}$ and $0 \not\equiv f \in R[X_1, \ldots, X_n]$. Suppose that f is irreducible and changes sign on R^n (i.e. there exist $\underline{x}, \underline{y} \in R^n$ with $f(\underline{x})f(\underline{y}) < 0$). Show that $\langle f \rangle = \mathcal{I}(\mathcal{Z}(f))$.

The bonus exercise is voluntary and will be awarded extra points. Please hand in your solutions by ***Thursday, 31 January 2019, 08:15h*** *(postbox 16 in F4).*