Fachbereich Mathematik und Statistik
Prof. Dr. Salma Kuhlmann
Lothar Sebastian Krapp
Simon Müller
WS 2018 / 2019

Universität Konstanz

Real Algebraic Geometry I

Exercise Sheet 13
Transcendence bases

Exercise 47
(4 points)

(a) Show that the ring of formal power series $\mathbb{R}[\![\underline{x}]\!]$ is local, i.e. it contains exactly one maximal ideal.

(b) Let $f \in \mathbb{R}[\![\underline{x}]\!]$ with

$$f = f_k + f_{k+1} + \ldots$$

where $k \in \mathbb{N}$ and each $f_i \in \mathbb{R}[\underline{x}]$ is homogeneous and of degree i. Suppose that f is sos in $\mathbb{R}[\![\underline{x}]\!]$. Show that k is even and f_k is a sum of squares of forms of degree $\frac{k}{2}$.

Exercise 48
(5 points)
Let $(K, \leq)$ be an ordered field.

(a) Let F be a subfield of K and let $a, b \in K$ with $a < b$. Suppose that every element in the interval $]a, b[\subseteq K$ is algebraic over F. Show that K is algebraic over F.

(b) Show that if $\mathrm{trdeg}(K) = \aleph_0$ over $\mathbb{Q}$, then the cardinality of K is $\aleph_0$.

(c) Suppose that $\mathrm{trdeg}(K) = \aleph_0$ over $\mathbb{Q}$. Show that there exists a transcendence basis $A = \{a_1, a_2, \ldots\}$ of $K/\mathbb{Q}$ which is dense in K, i.e. for any $c, d \in K$ with $c < d$ there is some $i \in \mathbb{N}$ such that $c < a_i < d$.

(Hint: Let $\mathcal{B} = \{I_j \mid j \in \mathbb{N}\}$ be the set of all open intervals $]a, b[$ in K. For every $I_j \in \mathcal{B}$, find a suitable $a_j \in I_j$.)

Exercise 49
(3 points)
Let $(K, \leq)$ be an ordered field and let R be the real closure of K. Show that for any set $A \subseteq K$ and any subfield $F \subseteq K$, the set A is a transcendence basis of K/F if and only if it is a transcendence basis of R/F.

Exercise 50
(4 points)
Let A be a commutative ring with 1 containing $\mathbb{Q}$. Let T be a generating preprime and M a maximal proper archimedean T-module. Show that the map $\alpha\colon A \to \mathbb{R}, a \mapsto \inf(\mathrm{cut}(a))$ is a ring homomorphism.
(This exercise requires the material convered in Lecture 27.)

Bonus Exercise
(4 points)
Show that the transcendence degree of $\mathbb{R}$ over $\mathbb{Q}$ is $2^{\aleph_0}$.

*The bonus exercise is voluntary and will be awarded extra points. Please hand in your solutions by **Monday, 11 February 2019, 11:45h** (postbox 16 in F4).*