Fachbereich Mathematik und Statistik
Prof. Dr. Salma Kuhlmann
Lothar Sebastian Krapp
Simon Müller
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Universität Konstanz

Real Algebraic Geometry I

Bonus Exercise Sheet
Merry Christmas!

Merry Christmas also to the class tutor! This bonus sheet is voluntary and will not be discussed in tutorials. However, you can earn bonus points by tackling these problems and handing in your solutions after the Christmas break.

Bonus Exercise 1
For any $r \in \mathbb{R}$, let r_2 denote its dyadic expansion, i.e. r_2 is a function from $\mathbb{Z}$ to $\{0, 1\}$ such that

$$r = \sum_{i=-\infty}^{\infty} r_2(i) 2^i.$$

Define the relation $\preceq$ on $\mathbb{R}$ as follows: For any $r, s \in \mathbb{R}$,

$$r \preceq s :\Leftrightarrow \forall i \in \mathbb{Z} : r_2(i) \leq s_2(i).$$

(a) Show that $\preceq$ defines a partial order on $\mathbb{R}$ which is not total.

(b) Is $\preceq$ compatible with addition, i.e. is it true that for any $a, b, c \in \mathbb{R}$ with $a \preceq b$ we have $a + c \preceq b + c$?

(c) Is $\preceq$ compatible with multiplication, i.e. is it true that for any $a, b, c \in \mathbb{R}$ with $a \preceq b$ and $0 \preceq c$ we have $ac \preceq bc$?

(d) Is $\mathbb{Q}$ dense in $\mathbb{R}$ with respect to $\preceq$, i.e. is it true that for any $a, b \in \mathbb{R}$ with $a \prec b$ there is some $c \in \mathbb{Q}$ with $a \preceq c \preceq b$?

(e) How else could you describe this partial order on $\mathbb{R}$ without referring to the dyadic expansion?

Bonus Exercise 2
Show that for any infinite set A, there is a real closed field R such that A and R have the same cardinality.

Bonus Exercise 3
Recall that by Hölder's Theorem any archimedean ordered field can be embedded into $\mathbb{R}$. Since $\mathbb{R}$ is real closed, we can say that $\mathbb{R}$ is a universal domain for all archimedean real closed fields. In the following, we will introduce the class of **surreal numbers No**, which can be considered as a universal domain for *all* real closed fields.
We call a a pair $(A, \leq)$ of a set A and a relation $\leq$ on A a **well-ordering** if $\leq$ is a total order on A such that any non-empty subset of A has a minimum with respect to $\leq$. Fix a class of representatives for all order types of well-orderings and denote it by **On**. In other words, every element of **On** is a well-ordering, and for any well-ordering $(A, \leq)$ there is an element $(B, \leq') \in \mathbf{On}$ such that there is an order-preserving bijection from $(A, \leq)$ to $(B, \leq')$. We call an element of **On** an **ordinal number**.
Let **No** be the following proper class:

$$\mathbf{No} = \{s : \alpha \to \{-1, 1\} \mid (\alpha, \leq) \in \mathbf{On}\}.$$

No can be equipped with the operations of addition and multiplication as well as an ordering $\leq_{\text{lex}}$ such that it satisfies the axioms of an ordered field (except that the underlying class is not a set but a proper class). The ordered field of surreal numbers $(\mathbf{No}, \leq)$ is also called a universal domain for all real closed fields, as **No** is real closed and any real closed field (whose underlying class is a set) can be embedded into $(\mathbf{No}, \leq)$.
In this exercise, we want to verify some set and order properties of **No**.

(a) Show that for any set A, the cardinality of **No** is greater than the cardinality of A. Hence or otherwise, show that **No** is a proper class.

(Hint: You may assume that any set can be well-ordered. This is called the ***well-ordering principle****, which is equivalent to the axiom of choice.)*

(b) For any $n \in \mathbb{N}_0$, let $\mathbf{n} = \{0, \dots, n-1\}$ be equipped with its standard ordering. Note that $\mathbf{0} = \emptyset$. Let

$$\mathbf{No}_\omega = \{s : \mathbf{n} \to \{-1, 1\} \mid n \in \mathbb{N}_0\}.$$

Let $a, b \in \mathbf{No}_\omega$ and let $n, m \in \mathbb{N}_0$ such that the domain of a is $\mathbf{n}$ and the domain of b is $\mathbf{m}$. We say that a **is simpler than** b if $n \leq m$ and $b(i) = a(i)$ for all $i \in \mathbf{n}$. We write $a \leq_{\text{s}} b$.

Show that $\leq_{\text{s}}$ defines a partial ordering on $\mathbf{No}_\omega$ which is not total.

(c) Define the following relation $<_{\text{lex}}$ on $\mathbf{No}_\omega$:

$$a <_{\text{lex}} b :\Leftrightarrow a(i_0) < b(i_0) \text{ for } i_0 = \min\{i \in \mathbb{N}_0 \mid a(i) \neq b(i)\},$$

where we set $a(i) = 0$ (respectively $b(i) = 0$) for any $i \in \mathbb{N}_0$ which is not in the domain of a (respectively of b).

Show that $\leq_{\text{lex}}$ is well-defined and that it defines a total order on $\mathbf{No}_\omega$. Find some elements $a, b \in \mathbf{No}_\omega$ such that $a \leq_{\text{lex}} b$ but $b \leq_{\text{s}} a$.

(d) Let $a \in \mathbf{No}_\omega$ with domain $\mathbf{n}$. We also denote a by a sign sequence representation consisting of the symbols $\oplus$ and $\ominus$ as follows: If $a(i) = 1$, then the i^{th} digit of the sign sequence representation is $\oplus$, otherwise it is $\ominus$. For instance, the function $a : \{0, 1, 2\} \to \{-1, 1\}$ given by $a(0) = a(2) = 1$ and $a(1) = -1$ is denoted by $a = \oplus \ominus \oplus$.

Compare the elements $\oplus\oplus$, $\ominus \oplus \ominus$, $\oplus$, $\oplus \ominus \ominus$, $\oplus\ominus$, $\oplus \ominus \oplus$ and $\ominus\ominus$ by $\leq_{\text{lex}}$ and $\leq_{\text{s}}$.

Bonus Exercise 4

(a) Let $(\Gamma_1, \leq_1)$ and $(\Gamma_2, \leq_2)$ be totally ordered sets and $\Gamma = \Gamma_1 \times \Gamma_2$. Define the relation $\leq_{\text{lex}}$ on Γ by

$$(a, b) \leq_{\text{lex}} (c, d) :\Leftrightarrow [a <_1 c \vee (a = c \wedge b \leq_2 d)].$$

Show that $\leq_{\text{lex}}$ defines a total ordering on Γ. This is called the **lexicographic ordering**.

(b) Let $n \in \mathbb{N}$ and $(\mathbb{N}, \leq)$ where $\leq$ denotes the standard ordering on $\mathbb{N}$. Show that

$$\mathbb{N}^n = \underbrace{\mathbb{N} \times \ldots \times \mathbb{N}}_{n \text{ times}}$$

is well-ordered under the lexicographic ordering.

The order type of $(\mathbb{N}^n, \leq_{\text{lex}})$ is denoted by ω^n.

(c) Let $H = \{s : -\mathbb{N} \to \mathbb{N} \mid \operatorname{supp}(s) \text{ is finite}\}$, where $\operatorname{supp}(s) = \{i \in -\mathbb{N} \mid s(i) \neq 0\}$. Define the following relation on H:

$$s < r :\Leftrightarrow s \neq r \wedge (r - s)(\min \operatorname{supp}(r - s)) > 0.$$

Show that $(H, \leq)$ is a totally ordered set. Show further that $\leq$ defines a well-ordering on H.

The order type of $(H, \leq)$ is denoted by ω^ω.

(d) Let $(A, \leq)$ be a totally ordered set with order type ω^n for some $n \in \mathbb{N}$ and let $(B, \leq)$ be a totally ordered set with order type ω^ω. Show that there is an order-preserving injection $f : A \to B$ such that $f(A)$ is a proper initial segment of B, i.e. $f(A) \subsetneq B$ and for any $a \in A$ we have

$$\{b \in B \mid b \leq f(a)\} = \{f(x) \mid x \in A \text{ with } x \leq a\}.$$

Please hand in your solutions by ***Thursday, 10 January 2019, 08:15h*** *(postbox 16 in F4).*