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WS 2018 / 2019

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Real Algebraic Geometry I

Revision Sheet

Revision Exercise 1
Recall that Schmüdgen's Positivstellensatz does not hold for non-archimedean real closed fields (see Lecture 23, Remark 2.2.5). By going through the proofs in detail, verify that Schmüdgen's Positivstellensatz holds for arbitrary Archimedean real closed fields (see Lecture 23, Remark 2.2.6) and that the Abstract Positivstellensatz (see Lecture 19, Theorem 3.7) and the Positivstellensatz (see Lecture 17, Theorem 1.1) hold for arbitrary real closed fields.

Revision Exercise 2
Let A be a commutative ring with 1 and let $\chi = \mathrm{Hom}(A, \mathbb{R})$. Consider the map $\varphi\colon\ \chi \to \mathrm{Sper}(A), \alpha \mapsto P_\alpha = \alpha^{-1}\left(\mathbb{R}^{\geq 0}\right)$.

(a) Show the following:

- (i) φ is well-defined, i.e. P_α is an ordering of A for any $\alpha \in \chi$.
- (ii) φ is injective.
- (iii) For any $\alpha \in \chi$, we have $\mathrm{supp}(P_\alpha) = \ker(\alpha)$.

(b) For every $a \in A$, define

$$\hat{a}\colon\ \chi \to \mathbb{R}, \alpha \mapsto \hat{a}(\alpha) := \alpha(a)$$

and

$$u(\hat{a}) = \{\alpha \in \chi \mid \hat{a}(\alpha) > 0\}.$$

Show the following:

- (i) The collection $\mathcal{B} = \{u(\hat{a}) \mid a \in A\}$ is a sub-base for a topology τ on χ.
- (ii) For every $a \in A$, the map $\hat{a}$ is continuous with respect to the topology τ.
- (iii) If τ' is another topology on χ such that $\hat{a}$ is continuous for every $a \in A$, then $u(\hat{a}) \in \tau'$ for every $a \in A$.
- (iv) If we endow $\mathrm{Sper}(A)$ with the spectral topology, then the topology on χ induced by φ coincides with τ.

Revision Exercise 3
Let S and S' be the subsets of $\mathbb{R}[X]$ given by $S = \{1 - X, 1 + X\}$, $S' = \{1 - X^2\}$, and let $K = [-1, 1] \subseteq \mathbb{R}$.

(a) Show that $K = K_S = K_{S'}$.

(b) Show that T_S and $T_{S'}$ are saturated.

Revision Exercise 4
Let $(K, \leq)$ be an ordered field.

(a) Define a relation $\sim$ on K by $a \sim b$ if and only if there is some $n \in \mathbb{N}$ such that $|a| < n|b|$ and $|b| < n|a|$. Show that $\sim$ defines an equivalence relation on K.

(b) Let $G = \{[a] \mid a \in K \setminus \{0\}\}$, i.e. the set of equivalence classes of $K \setminus \{0\}$ under $\sim$. Let v be the natural valuation on K, i.e. the map $v\colon\ K \to G \cup \{\infty\}, a \mapsto [a]$, where ∞ stands for $[0]$. Show that G is a group under addition given by $[a] + [b] = [ab]$. Show that $(G \cup \{\infty\}, \leq)$ is a totally ordered set, where the order relation is given by

$$[a] \leq [b] :\Longleftrightarrow [a] = [b] \vee |b| < |a|.$$

(c) Set $0 = [1]$. Let $\theta_v = \{a \in K \mid v(a) \geq 0\}$ and let $\mathcal{I}_v = \{a \in K \mid v(a) > 0\}$. Show that θ_v is a ring and that $\mathcal{I}_v$ is a maximal ideal of θ_v.

(d) Let $\overline{K} = \theta_v/\mathcal{I}_v$. Define the induced ordering $\leq$ on $\overline{K}$ by

$$\overline{a} \leq \overline{b} :\Longleftrightarrow \overline{a} = \overline{b} \vee a < b.$$

Show that $(\overline{K}, \leq)$ is an Archimedean ordered field.

This problem sheet is for your personal revision. Solutions will not be marked.