Fachbereich Mathematik und Statistik
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Universität Konstanz

Real Algebraic Geometry II

Exercise Sheet 1
Valued modules

Let Z be a commutative ring with 1. All modules we consider are left Z-modules.

Exercise 1
(4 points)
Let (M, v) be a valued module.

(a) Show that M is torsion-free.

(b) Show that for any $x, y \in M$, the following hold:

(i) $v(-x) = v(x)$,

(ii) $v(x) \neq v(y) \;\Rightarrow\; v(x+y) = \min\{v(x), v(y)\}$,

(iii) $v(x+y) > v(x) \;\Rightarrow\; v(x) = v(y)$.

Exercise 2
(4 points)
Let $v\colon\ Z[x] \to \mathbb{N}_0 \cup \{\infty\}$ be given by $v(0) = \infty$ and $v(p) = \min\{k \mid a_k \neq 0\}$ for $p(x) = \sum_{k=0}^{n} a_k x^k \in Z[x] \setminus \{0\}$.

(a) Suppose that $Z = \mathbb{Z}$.

(i) Show that $(Z[x], v)$ is a valued module.

(ii) Determine the skeleton of $(Z[x], v)$. Hence, or otherwise, find a Hahn sum $\coprod_{\gamma\in\Gamma} B(\gamma)$ such that

$$(Z[x], v) \cong \left(\coprod_{\gamma\in\Gamma} B(\gamma), v_{\min} \right).$$

(b) Does (i) also hold when Z is an arbitrary commutative ring with 1? Justify your answer!

Exercise 3
(4 points)
Let (M_1, v_1) and (M_2, v_2) be two valued modules with value sets $\Gamma_1 = v_1(M_1)$ and $\Gamma_2 = v_2(M_2)$. Moreover, let $h\colon M_1 \to M_2$ be an isomorphism of Z-modules which preserves the valuation.

(i) Let $\widetilde{h}\colon \Gamma_1 \to \Gamma_2, v_1(x) \mapsto v_2(h(x))$. Show that $\widetilde{h}$ is well-defined and an isomorphism of ordered sets, i.e. an order-preserving bijection from Γ_1 to Γ_2.

(ii) Show that for each $\gamma \in \Gamma_1$, the map h_γ given by

$$B(M_1, \gamma) \to B\left(M_2, \widetilde{h}(\gamma)\right), \ \pi^{M_1}(\gamma, x) \mapsto \pi^{M_2}\left(\widetilde{h}(\gamma), h(x)\right).$$

is an isomorphism of Z-modules.

Exercise 4
(4 points)
Let $[\Gamma, \{B(\gamma) \mid \gamma \in \Gamma\}]$ be an ordered system of torsion-free modules.

(i) Show that $\mathbf{H}_{\gamma\in\Gamma} B(\gamma)$ is a module and that $\coprod_{\gamma\in\Gamma} B(\gamma)$ is a submodule of $\mathbf{H}_{\gamma\in\Gamma} B(\gamma)$.

(ii) Show that $S\left(\coprod_{\gamma\in\Gamma} B(\gamma)\right) \cong [\Gamma, \{B(\gamma) \mid \gamma \in \Gamma\}] \cong S\left(\mathbf{H}_{\gamma\in\Gamma} B(\gamma)\right)$.

Please hand in your solutions by ***Thursday, 25 April 2019, 10:00h*** *(postbox 14 in F4).*