Fachbereich Mathematik und Statistik
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## Real Algebraic Geometry II

## Exercise Sheet 3
## Valuation independence

Let $Q$ be a field. If not further specified, any vector space we consider is a $Q$-vector space.

**Exercise 7**
**(4 points)**
Recall that the polynomial ring $\mathbb{R}[x]$ is a subring of the ring of formal power series $\mathbb{R}[\![x]\!]$. Consider both of these as $\mathbb{R}$-vector spaces.
Let $v$ be the valuation on $\mathbb{R}[x]$ given in Exercise 2.

(a) Show that $v$ extends to a valuation $v_1$ on $\mathbb{R}[\![x]\!]$, i.e. that there exists a valuation $v_1$ on $\mathbb{R}[\![x]\!]$ with $v_1(p) = v(p)$ for any $p \in \mathbb{R}[x]$, such that the extension

$$(\mathbb{R}[x], v) \subseteq (\mathbb{R}[\![x]\!], v_1)$$

is immediate.

(b) Find an extension $v_2$ of $v$ to $\mathbb{R}[\![x]\!]$ such that

$$(\mathbb{R}[x], v) \subseteq (\mathbb{R}[\![x]\!], v_2)$$

is not immediate.

**Exercise 8**
**(4 points)**
Let $(V_1, v_1)$ and $(V_2, v_2)$ be valued vector spaces such that $S(V_1) = S(V_2)$. Let $h\colon V_1 \to V_2$ be a valuation preserving isomorphism of vector spaces, i.e. for any $a \in V_1$, we have $v_2(h(a)) = v_1(a)$. Let $\mathcal{B} \subseteq V_1 \setminus \{0\}$.

(a) Show that $\mathcal{B}$ is $Q$-valuation independent if and only if $h(\mathcal{B})$ is $Q$-valuation independent.

(b) Show that $\mathcal{B}$ is a $Q$-valuation basis for $(V_1, v_1)$ if and only if $h(\mathcal{B})$ is a $Q$-valuation basis for $(V_2, v_2)$.

**Exercise 9**
**(4 points)**
Let $(V_1, v_1)$ and $(V_2, v_2)$ be valued vector spaces such that $S(V_1) = S(V_2)$. Let $\mathcal{B}_1 \subseteq V_1 \setminus \{0\}$ be a $Q$-valuation basis for $(V_1, v_1)$ and let $\mathcal{B}_2 \subseteq V_2 \setminus \{0\}$ be a $Q$-valuation basis for $(V_2, v_2)$. Suppose that there exists a valuation preserving bijection

$$\widetilde{h}\colon\ \mathcal{B}_1 \to \mathcal{B}_2.$$

Let $h\colon\ V_1 \to V_2$ be the isomorphism obtained by linearly extending $\widetilde{h}$.
Show that $h$ is valuation preserving.

**Exercise 10**
**(4 points)**
Consider the $\mathbb{Q}$-vector space $(V, v) = (\mathbf{H}_{n\in\mathbb{N}} B_n, v_{\min})$.

(a) Let $B_n = \mathbb{Q}$ for any $n \in \mathbb{N}$.

  (i) Describe a maximal $\mathbb{Q}$-valuation independent set $\mathcal{B} \subseteq V$ such that support$(b)$ is a singleton for any $b \in \mathcal{B}$.

  (ii) Describe a maximal $\mathbb{Q}$-valuation independent set $\mathcal{B} \subseteq V$ such that support$(b)$ is infinite for any $b \in \mathcal{B}$.

(b) Let $B_n = \mathbb{R}$ for any $n \in \mathbb{N}$.

  (i) Describe a maximal $\mathbb{Q}$-valuation independent set $\mathcal{B} \subseteq V$ such that support$(b)$ is a singleton for any $b \in \mathcal{B}$.

  (ii) Describe a maximal $\mathbb{Q}$-valuation independent set $\mathcal{B} \subseteq V$ such that support$(b)$ is infinite for any $b \in \mathcal{B}$.

(c) *Bonus exercise:* For any $n \in \mathbb{N}$, let $B_{2n-1} = \mathbb{R}$ and $B_{2n} = \mathbb{Q}$. Describe a maximal $\mathbb{Q}$-valuation independent set $\mathcal{B} \subseteq V$ such that support$(b)$ is infinite for any $b \in \mathcal{B}$.

*The bonus exercise is voluntary and will be awarded extra points. Please hand in your solutions by* ***Thursday, 09 May 2019, 10:00h*** *(postbox 14 in F4).*