Fachbereich Mathematik und Statistik
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Universität Konstanz

Real Algebraic Geometry II

Exercise Sheet 5
Pseudo-completeness and valued groups

Exercise 15
(4 points)
Let Q be a field and let (V_1, v_1) and (V_2, v_2) be Q-valued vector spaces. Let $h\colon V_1 \to V_2$ be a valuation preserving isomorphism and let $S = \{a_\rho\}_{\rho<\lambda}$ be a pseudo-Cauchy sequence in (V_1, v_1).

(a) Show that $h(S) = \{h(a_\rho)\}_{\rho<\lambda}$ is a pseudo-Cauchy sequence in (V_2, v_2).

(b) Let x be a pseudo-limit of S in V_1. Show that $h(x)$ is a pseudo-limit of $h(S)$ in V_2.

(c) Deduce that (V_1, v_1) is pseudo-complete if and only if (V_2, v_2) is pseudo-complete.

Exercise 16
(4 points)
Let Q be a field and let (V, v) be a Q-valued vector space. Let $S = \{a_\rho\}_{\rho<\lambda}$ be a pseudo-Cauchy sequence in (V, v) with pseudo limit $s \in V$. Let $q \in Q \setminus \{0\}$ and let $x \in V$.

(a) Show that $qS = \{qa_\rho\}_{\rho<\lambda}$ is pseudo-Cauchy with pseudo-limit qs.

(b) Show that $x + S = \{x + a_\rho\}_{\rho<\lambda}$ is pseudo-Cauchy with pseudo-limit $x + s$.

(c) Suppose that 0 is a pseudo-limit of $x + qS = \{x + qa_\rho\}_{\rho<\lambda}$. Show that $-\frac{x}{q}$ is a pseudo-limit of S.

(d) Let $T = \{b_\rho\}_{\rho<\lambda}$ be a pseudo-Cauchy sequence in (V, v) with pseudo-limit $t \in V$. Is

$$\{a_\rho + b_\rho\}_{\rho<\lambda}$$

necessarily pseudo-Cauchy with pseudo-limit $s + t$? Justify your answer!

Exercise 17
(4 points)
Let $p \in \mathbb{N}$ be a prime and let $(G, +, 0)$ be an abelian p-group, i.e. an abelian group such that for any $g \in G$ there exists $n \in \mathbb{N}$ with $p^n g = 0$. Suppose that

$$\bigcap_{n \in \mathbb{N}} p^n G = \{0\}.$$

The height function h on G is given by

$$h\colon\ g \mapsto \begin{cases} n & \text{if } g \in p^n G \setminus p^{n+1} G \text{ for some } n \in \mathbb{N}_0, \\ \infty & \text{otherwise.} \end{cases}$$

Show that (G, h) is a valued group.

Exercise 18
(4 points)
Let $(G, +, 0, <)$ be an ordered abelian group.

(a) Show that $\sim^+$ is an equivalence relation on G.

(b) Let $x, y, z \in G \setminus \{0\}$ such that $x \ll^+ y$.

  (i) Suppose that $z \sim^+ x$. Show that $z \ll^+ y$.
  (ii) Suppose that $z \sim^+ y$. Show that $x \ll^+ z$.

(c) Deduce that $(\Gamma, <_\Gamma)$ is totally ordered.

*Please hand in your solutions by **Thursday, 23 May 2019, 10:00h** (postbox 14 in F4).*