Fachbereich Mathematik und Statistik
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Universität Konstanz

Real Algebraic Geometry II

Exercise Sheet 6
Ordered abelian groups

Exercise 19
(5 points)
Let G be an ordered abelian group. Let $C \subseteq G$ be a convex subgroup and $B = G/C$.

(a) Define the relation $<_B$ on B by

$$g_1 + C <_B g_2 + C \quad :\Longleftrightarrow \quad (g_2 - g_1 \notin C \wedge g_2 - g_1 > 0)$$

for any $g_1, g_2 \in G$. Show that $(B, +, 0, <_B)$ is an ordered abelian group.

(b) Show that the set of convex subgroups of G is totally ordered by the relation $\subseteq$.

(c) Find a bijective correspondence between convex subgroups of B and convex subgroups $C' \subseteq G$ with $C \subseteq C'$.

(d) Let D_1 and D_2 be convex subgroups of G such that $D_1 \subseteq D_2$ and there are no further convex subgroups between D_1 and D_2. Show that D_2/D_1 has no non-trivial convex subgroups.

(e) Show that G is Archimedean if and only if its only convex subgroups are $\{0\}$ and G.

Exercise 20
(3 points)
Let G be an ordered abelian group and let $x \in G \setminus \{0\}$.

(a) Show that C_x and D_x are convex subgroups of G with $D_x \subsetneq C_x$.

(b) Show that D_x is the largest proper convex subgroup of C_x (with respect to the linear ordering given by $\subseteq$).

(c) Deduce that the ordered abelian group C_x/D_x is Archimedean.

Exercise 21
(4 points)
Let G be an ordered abelian group.

(a) Let v be defined as in Lecture 9, Proposition 3.5. Show that v is a valuation on G, i.e. that (G, v) is a valued $\mathbb{Z}$-module.

(b) Let $x \in G \setminus \{0\}$. Show that

$$G^{v(x)} = \bigcap\{C \mid C \text{ is a convex subgroup of } G \text{ and } x \in C\}$$

and

$$G_{v(x)} = \bigcup\{C \mid C \text{ is a convex subgroup of } G \text{ and } x \notin C\}.$$

Conclude that $B_x = B(G, v(x))$ and that B_x is an Archimedean.

Exercise 22
(4 points)
Let $[\Gamma, \{B(\gamma) \mid \gamma \in \Gamma\}]$ be an ordered family of Archimedean ordered abelian groups. Let

$$G = \coprod_{\gamma \in \Gamma} B(\gamma)$$

and define a relation $<_{\text{lex}}$ on G by

$$0 <_{\text{lex}} g :\Longleftrightarrow (g \neq 0 \wedge g(v_{\min}(g)) > 0).$$

(a) Show that $(G, <_{\text{lex}})$ is an ordered abelian group.

(b) Show that $v_{\min}$ and the natural valuation v on G are equivalent.

Please hand in your solutions by ***Friday, 31 May 2019, 10:00h*** *(postbox 14 in F4).*