Fachbereich Mathematik und Statistik
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Universität Konstanz

Real Algebraic Geometry II

Exercise Sheet 7
Valued fields

Exercise 23
(4 points)
Let $p \in \mathbb{N}$ be prime. Define the map v_p on $\mathbb{Q}$ as follows:

- Let $v_p(0) = \infty$.
- For any $k \in \mathbb{Z} \setminus \{0\}$, let $v_p(k) = \max\Big\{\ell \in \mathbb{N}_0 \;\Big|\; p^\ell \text{ divides } k\Big\}$.
- For any $k, m \in \mathbb{Z} \setminus \{0\}$, let $v_p\Big(\frac{k}{m}\Big) = v_p(k) - v_p(m)$.

(a) Show that v_p is a valuation on $\mathbb{Q}$.

(b) Determine R_{v_p}, I_{v_p}, U_{v_p} and K_{v_p}.

Exercise 24
(4 points)
Let $K = \mathbb{R}(\!(X)\!)$.

(a) Let $v_{\min}$ be the map on K defined in Real Algebraic Geometry I, Lecture 24, Notation 1.4. Show that $(K, v_{\min})$ is a valued field and determine its value group $G_{v_{\min}}$ and its residue field $K_{v_{\min}}$.

(b) Consider K as an ordered field with the ordering induced by $X < |r|$ for any $r \in \mathbb{R} \setminus \{0\}$. Let v be the natural valuation on K. Determine the value group G_v and the residue field K_v.

(c) Show that
$$\varphi\colon\ G_{v_{\min}} \to G_v, v_{\min}(x) \mapsto v(x)$$
is an order-preserving isomorphism of groups and that
$$\psi\colon\ K_{v_{\min}} \to K_v, a v_{\min} \mapsto a v$$
is an order-preserving isomorphism of fields.

Please hand in your solutions by ***Thursday, 06 June 2019, 10:00h*** *(postbox 14 in F4).*