Fachbereich Mathematik und Statistik
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Universität Konstanz

Real Algebraic Geometry II

Exercise Sheet 8
Hardy fields and Neumann's Lemma

Exercise 25
(4 points)
Let H be a Hardy field.

(a) Recall the definition of the asymptotic equivalence relation $\sim$ on H (Lecture 14, Definition 2.1). Show that $\sim$ coincides with the Archimedean equivalence relation on H.

(b) Hence, or otherwise, show that $(v(H \setminus \{0\}), +, <)$ is an ordered abelian group and that v is a valuation on H.

(c) Show that

$$R_v = \left\{ f \in H \;\middle|\; \lim_{x\to\infty} f(x) \in \mathbb{R} \right\},$$
$$I_v = \left\{ f \in H \;\middle|\; \lim_{x\to\infty} f(x) = 0 \right\} \qquad \text{and}$$
$$\mathcal{U}_v = \left\{ f \in H \;\middle|\; \lim_{x\to\infty} f(x) \in \mathbb{R}^\times \right\}.$$

Exercise 26
(4 points)
Let G be an ordered abelian group. Let $A, B \subseteq G$ be non-empty and well-ordered subsets. Show that

$$A + B = \{a + b \mid (a, b) \in A \times B\}$$

is a well-ordered subset of G.

Exercise 27
(4 points)
Let k be an Archimedean field and let G be an ordered abelian group.

(a) Show that $<_{\text{lex}}$ is a field ordering on $k(\!(G)\!)$, i.e. that for any $a, b, c \in k(\!(G)\!)$ we have

- if $a <_{\text{lex}} b$, then $a + c <_{\text{lex}} b + c$;
- if $0 <_{\text{lex}} a$ and $0 <_{\text{lex}} b$, then $0 <_{\text{lex}} ab$.

(b) Let $\varepsilon \in k(\!(G)\!)$ with $\text{support}(\varepsilon) \subseteq G^{>0}$. Show that

$$\sum_{n=0}^{\infty} \varepsilon^n \in k(\!(G)\!) \quad \text{and} \quad (1-\varepsilon)\left(\sum_{n=0}^{\infty} \varepsilon^n\right) = 1.$$

(c) Let $g_1, g_2 \in G$. Compute $(t^{g_1} + t^{g_2})^{-1}$.

Exercise 28
(4 points)
Let G be a divisible ordered abelian group and let $K = \mathbb{R}(\!(G)\!)$. For any $\varepsilon \in I_v$ define

$$e(\varepsilon) = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!}.$$

(a) Show that e is a well-defined function from I_v to $1 + I_v$.

(b) Show that e is an order-preserving homomorphism from $(I_v, +, 0, <)$ to $(1 + I_v, \cdot, 1, <)$.

(c) *Bonus exercise:* Show that

$$\ell\colon\ 1 + I_v \to I_v, 1 + \varepsilon \mapsto \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\varepsilon^n}{n}$$

is the inverse function of e.

*The bonus exercise is voluntary and will be awarded extra points. Please hand in your solutions by **Thursday, 13 June 2019, 10:00h** (postbox 14 in F4).*