Fachbereich Mathematik und Statistik
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SoSe 2019

Universität Konstanz

Real Algebraic Geometry II

Exercise Sheet 11
Integer parts and convex valuations

Exercise 33
(4 points)
Let K be an ordered field and let Z be an integer part of K.

(a) Show that for any $x \in K$, there exists a *unique* $z_x \in Z$ with

$$z_x \leq x < z_x + 1.$$

(b) Show that $\mathrm{ff}(Z)$ is dense in K.

Exercise 34
(3 points)

(a) Let K be an ordered field. Show that K is Archimedean if and only if $\mathbb{Z}$ is its unique integer part.

(b) Find an ordered field K and an integer part Z of K such that for any $n, m \in \mathbb{N}$, the polynomial $X^n - m$ has a root in $\mathrm{ff}(Z)$. Can K be Archimedean? Justify your answer!

Exercise 35
(5 points)
Let K be a field with valuations w_1 and w_2.

(a) Show that the following are equivalent:

  (i) w_2 is coarser than w_1.

  (ii) $I_{w_2} \subseteq I_{w_1}$.

  (iii) For any $a, b \in K$, if $w_1(a) \leq w_1(b)$, then $w_2(a) \leq w_2(b)$.

(b) Suppose that w_2 is coarser than w_1. Let

$$\varphi\colon\ K_{w_2} \to Kw_2, a \mapsto aw_2$$

be the residue map of w_2, where K_{w_2} denotes the valuation ring and Kw_2 the residue field of (K, w_2). Show that $\varphi(K_{w_1})$ is a valuation ring of the residue field Kw_2.

Exercise 36
(4 points)

(a) Let $\mathbb{K} = \mathbb{R}((\mathbb{Q} \times \mathbb{R}))$, where $\mathbb{Q} \times \mathbb{R}$ is ordered lexicographically. Let

$$C = \{(0, z) \mid z \in \mathbb{R}\}.$$

(i) Compute the convex valuation w on $\mathbb{K}$ associated to C.

(ii) Find the value group and the residue field of $(\mathbb{K}, w)$.

(iii) Compute the rank of $\mathbb{K}$.

(b) Let $K = \mathbb{R}(t)$. Show that for any ordering on K the rank of K is a singleton with $\mathcal{R} = \{K\}$.

*Please hand in your solutions by **Thursday, 04 July 2019, 10:00h** (postbox 14 in F4).*