# Real Algebraic Geometry and Model Theory

Universität Konstanz

# Abstracts

Françoise Point, FNRS-FRS-UMons

## Nullstellensatz in exponential fields

**Abstract** We will first recall the construction of free exponential rings, the notion of exponential ideals and the closure operators introduced by Macintyre, Wilkie and Kirby. Then we will discuss a weak nullstellensatz in exponential fields. This is work in progress with Nathalie Regnault.

Nikolaas Verhulst, TU Dresden

## Generalizing a result of Shtipel'man

**Abstract** In this talk, an old valuation theoretical proof of Makar-Limanov will be dissected. We will isolate the different semi-independent parts that make the whole tick and cut away all superfluous specificity. This will allow us to generalise Shtipel'man's theorem about valuations on Weyl-fields.

Andre Opris, Universität Passau

## Tamm's Theorem for log-analytic functions

**Abstract** In the paper "Extending Tamm's Theorem" by L. van den Dries and C. Miller a parametric result of Tamm's theorem is given. This talk is about a generalisation of this result: Let $X \subseteq \mathbb{R}^{n+m}$ and $f : X \to \mathbb{R}$ be a log-analytic function. This means that $f$ is a global subanalytic function augmented by logarithmic terms. Then there exists some $N \in \mathbb{N}$, so that for all $(x_0, y_0) \in \mathbb{R}^{n+m}$, if $y \mapsto f(x_0, y)$ is $C^N$ in a neighbourhood of $y_0$, then $y \mapsto f(x_0, y)$ is real analytic in a neighbourhood of $y_0$. We also observe that this theorem doesn't hold in the o-minimal structure $\mathbb{R}_{\mathrm{an,exp}}$ in general.

Mickaël Matusinski, Institut de Mathématiques de Bordeaux

## Omega-fields and exponentiation

**Abstract** Joint work with A. Berarducci, S. Kuhlmann and V. Mantova. We introduce the notion of fields endowed with an omega-map, a particular case being Conway's omega exponentiation in the context of surreal numbers. We investigate these fields and their interaction with possible real exponential structure, in particular with explicit constructions in the context of Kuhlmann-Shelah's kappa-bounded generalized series fields.

Christoph Schulze, Universität Konstanz

## Cones of non-negative Polynomials

**Abstract** Let $n, d \in \mathbb{N}$ and $\mathbb{R}[\underline{X}]_{2d} := \mathbb{R}[X_0, X_1, \ldots, X_n]_{2d}$ be the vector space of homogeneous polynomials of degree $2d$. We denote by $\mathcal{P} \subseteq \mathbb{R}[\underline{X}]_{2d}$ the subset of polynomials which are non-negative on the real projective space $\mathbb{P}^n(\mathbb{R})$ (or more generally on some subset $S \subseteq \mathbb{P}^n(\mathbb{R})$). Then $\mathcal{P}$ is convex and closed under multiplication with $\mathbb{R}^+$ - therefore $\mathcal{P}$ is a convex cone.
The aim of this talk is to give an overview about results concerning the convex structure of $\mathcal{P}$. This includes a complete characterization of the facial structure for very small $n$ and $d$, results about extreme rays of $\mathcal{P}$ (which may be seen as the generating points of the cone), connections between faces of $\mathcal{P}$ and their zero set, relations to interpolation problems and information about faces of low codimension.

Tobias Kuna, University of Reading

## The infinite dimensional moment problem using projective limit techniques

**Abstract** The classical moment problem on $\mathbb{R}$ considers the question whether to a given sequence of numbers $m_1, m_2, \ldots$ one can find a probability $\mu$ such that $\int_{\mathbb{R}} x^k \mu(\mathrm{d}x) = m_k$ for $k \in \mathbb{N}$. One can reformulate the problem in a more abstract way: let $A$ be the real algebra of all polynomials on $\mathbb{R}$ and introduce the linear function $L : A \to \mathbb{R}$ such that the polynomial $p = \sum_k a_k X^k$ is mapped to $L(p) := \sum_k a_k m_k$. Then we see that the classical moment problem is a special instance of the following general version: when can a linear functional $L$ on a unital commutative real algebra $A$ be represented as an integral w.r.t. a measure on the character space of $A$. When the measure can be extended to the Borel $\sigma$–algebra generated by the weak topology and when the measure can be chosen to be Radon? In a joint work with Salma Kuhlmann, Maria Infusino and Patrick Michalski, we approach this problem by constructing the character space $X(A)$ as a projective limit of a certain family of finite dimensional analogues. Another natural measurability structure on $X(A)$ is the associated cylinder $\sigma$–algebra, which allows us to obtain representations of linear functionals, which are positive on sum of squares in $A$ and fulfil certain quasi-analytic bounds. The well-known Prokhorov theorem gives a criteria when the measure can be extended to the Borel $\sigma$–algebra. This generalize to infinitely (even uncountably) generated algebras some of the classical theorems for the moment problem such as the ones by Nussbaum and Putinar. Our results also apply in the case when $A$ is the algebra of polynomials in an arbitrary number of variables, providing alternative proofs of some recent results for this instance of the moment problem. Furthermore, our results offer at the same time a unified setting which enables comparisons to appreciate the difference between these results.

Siegfried Van Hille, KU Leuven

## Uniform parameterizations of bounded definable sets in $\mathbb{R}_{\mathrm{an}}^{\mathrm{pow}}$

**Abstract** If $X$ is a definable set in the structure $\mathbb{R}_{\mathrm{an}}^{\mathrm{pow}}$, then it follows easily from the cell decomposition theorem that there exists a finite collection of definable maps whose range cover X. We call this a parameterization. Moreover, if $X$ is bounded, one can ask these maps to have bounded derivatives up to order $r$, where $r$ is any positive integer. Recently, by work of Cluckers, Pila and Wilkie, it has been shown that one can take a parameterization such that the number of maps in such a parameterization grows polynomially in $r$. In a project for my PhD thesis, I try to control the degree of this polynomial in terms of the dimension of X. There is a general result in arbitrary dimensions and a sharp result in low dimensions. In fact, the result is true for a definable family and any suitable reduct of $\mathbb{R}_{\mathrm{an}}^{\mathrm{pow}}$ such as the semi-algebraic sets.

Gabriel Dill, Universität Basel

## Unlikely intersections between isogeny orbits and curves

**Abstract** In the spirit of the Mordell-Lang conjecture, we consider the intersection of a curve in a family of abelian varieties with the images of a finite-rank subgroup of a fixed abelian variety $A_0$ under all isogenies between $A_0$ and some member of the family. After excluding certain degenerate cases, we can prove that this intersection is finite if everything is defined over the algebraic numbers. To do so, we apply the Pila-Zannier method with a variant of the Pila-Wilkie theorem due to Habegger and Pila, counting "semi-rational" points on definable sets. This proves a slightly modified version of the so-called André-Pink-Zannier conjecture over the algebraic numbers in the case of curves. We can even allow translates of the finite-rank subgroup by abelian subvarieties of controlled dimension if we strengthen the degeneracy hypotheses suitably.

Pablo Andújar Guerrero, McMaster University

## Definable topological spaces in o-minimal structures

**Abstract** In model theory a topological space can be interpreted naturally as a second order structure, namely a set and the unary relation of subsets that corresponds to the topology. A different approach is that of a first order structure in which there exists a topological space with a basis that is (uniformly) definable. We call this a definable topological space. A natural example corresponds to the order topology on any linearly ordered structure.

During this talk I'll present results on definable topological spaces in an o-minimal structure $\mathcal{R} = (R, \ldots)$. In particular we will classify Hausdorff definable topological spaces $(X, \tau)$, where $X \subseteq R$. We'll consider a number of first order properties of these spaces that resemble topological properties and note how, in the o-minimal setting, the induced framework, which we might call "definable topology", resembles general topology. This is joint work with Margaret Thomas and Erik Walsberg.

Alexander Taveira Blomenhofer, Universität Konstanz

## A new Algorithm for Overcomplete Tensor Decomposition based on Sums-of-Squares Optimisation

**Abstract** Every symmetric tensor $T$ of degree $d$ may be represented as a linear combination $T = \sum_{i=1}^{m} \lambda_i \, a_i \otimes \ldots \otimes a_i$ of $d$-th tensor powers of vectors $a_i \in \mathbb{R}^n$, corresponding to linear forms $\langle a_i, X \rangle$. The task of finding such $a_i$ (when $T$ is given) is called the *tensor decomposition problem*. Tensor decomposition has a broad range of applications:

Symmetric tensors occur naturally e.g. as moment tensors of probability measures and tensor decomposition techniques can be used to find quadrature rules for them. However, tensor decomposition is also a computationally demanding task, particularly in the so-called overcomplete setting, where $m > n$. The approximation algorithms achieving the best known guarantees in this setting are based on the sums of squares (SOS) programming hierarchy, using the fact that symmetric tensors correspond to homogeneous polynomials, i.e. $\sum_{i=1}^{m} \lambda_i \, a_i \otimes \ldots \otimes a_i \;\longleftrightarrow\; \sum_{i=1}^{m} \lambda_i \langle a_i, X \rangle^d$.

We develop a new class of algorithms based on SOS programming that allow us to reduce a degree-$d$ homogeneous polynomial $T = \sum_{i=1}^{m} \langle a_i, X \rangle^d$ to (something close to) a rank-1 quadratic form via a reduction polynomial $W \in \sum \mathbb{R}[X]^2$ that can be thought of as a “weight function” attaining high values on merely one of the components $a_i$. The component can then be extracted by running an eigenvalue decomposition on the quadratic form $\sum_{i=1}^{m} W(a_i) \langle a_i, X \rangle^2$.