Generalising a result of Shtipel'man

Nikolaas D. Verhulst

TU Dresden

The mathematics lesson with

Dr. Nikolaas Verhulst

TU Dresden

Figure: The anatomy lesson with Dr. Nicolaes Tulp (from Wikipedia)

Shtipel'man's result

Shtipel'man's theorem

Every valuation on $\mathbb{D}_1(k)$ is abelian.

Abelian valuations

Definition

Let D be a skewfield and let Γ be a totally ordered group. A *valuation* is a surjective map $v : D \to \Gamma \cup \{\infty\}$ satisfying:

(V1) $v(x) = \infty \iff x = 0$

(V2) $v(xy) = v(x)v(y)$

(V3) $v(x + y) \leq \min\{v(x), v(y)\}$

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A valuation $v : D \to \Gamma$ is called *abelian* if $v(xy) = v(yx)$ for all x, y in D.

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Let k be a field. The *first Weyl field* $\mathbb{D}_1(k)$ is defined as the skewfield of fractions of $\mathbb{A}_1(k)$.

Makar-Limanov's proof, part 1

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- $$\left(\sum_{i,j} \alpha_{ij} x^i y^j\right) x = \sum_{i,j} \alpha_{ij} x^i y^{j-1}(xy-1) = \cdots = \\ = x \sum_{i,j} \alpha_{ij} x^i y^j - \sum_{i,j} j\alpha_{ij} x^i y^{j-1}$$

  so

  $$[x,-](r) = rx - xr = -\sum_{i,j} j\alpha_{ij} x^i y^{j-1}$$

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The vital organs

Lemma of the nilpotent Lie-bracket

Let R be a ring with skewfield of fractions D. Let $v : D \to \Gamma \cup \{\infty\}$ be a valuation on D. If $r \in R$ is such that $[r, -]$ is a nilpotent Lie-bracket, then $v(r) \in Z(\Gamma)$.

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One-dimension-for-free lemma

Let $R' \subseteq R$ be rings with skewfields of fractions $D' \subseteq D$ and let $v : D \to \Gamma \cup \{\infty\}$ be a valuation on D. Suppose $v|_{D'}$ is abelian and $\mathrm{GKdim}(R') = \mathrm{GKdim}(R) - 1$. Then v is abelian.

Playing with the organs, part 1

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Corollary

If R is a domain satisfying the Ore condition and where $[x, -]$ is nilpotent for every $x \in R$, then any valuation on the skewfield of fractions of R is abelian.

Playing with the organs, part 2

GKdim of Ore extensions

Suppose D is a skewfield with a finite dimensional generating subspace V. If σ is a $Z(D)$-automorphism, δ is a σ-derivation, and $\sigma(V) \subseteq V$ then

$$\mathrm{GKdim}(D[x;\sigma,\delta]) = \mathrm{GKdim}(D) + 1.$$

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Corollary

Suppose D is a skewfield with a finite dimensional generating subspace V. If σ is a $Z(D)$-automorphism, δ is a σ-derivation, $\sigma(V) \subseteq V$ and all valuations on D are abelian, then all valuations on $D[x,\sigma,\delta]$ are abelian.

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Hilbert fields are non-examples.

Thanks for your attention!

Questions?