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Algebraic and Model Theoretic Properties of O-minimal Exponential Fields

Lothar Sebastian Krapp

Universität Konstanz, Fachbereich Mathematik und Statistik, Schwerpunkt Reelle Geometrie und Algebra

sebastian.krapp@uni-konstanz.de, +49 (0) 7531 88 2592

Universität Konstanz

Abstract

My research project aims to investigate from both a model theoretic and an algebraic point of view properties of o-minimal exponential fields — a class of structures which has not yet been considered in the literature. Since the study of these structures is motivated by recent results on the the decidability question of the real exponential field, the properties under investigation aim to shed light on the conditions under which this decidability question can be answered positively.

Introduction

The *decidability problem of the real numbers* is the question whether there exists a general algorithm which determines about a given statement on the real numbers if the statement is true or false. A statement about the real numbers, in this sense, needs to be formed following certain rules. More specifically, it can only use the symbols 0, 1, $+$, $\cdot$, $<$, $=$, logical expressions such as "not", "and", "or", "if ... then", "for all" and "there exists" and variables such as x and y. For instance, the following would be a true statement about the real numbers: "For every x, if not $x = 0$ then $x \cdot x > 0$". Of course, also more complex statements can be formed, which can then, for example, describe how certain functions on the real numbers behave.

In 1948, Tarski solved the decidability problem of the real numbers by constructing the required algorithm (cf. [3]). He also posed the question whether there exists a similar algorithm if additionally the statements can use exponentiation. For example, "There exists x such that $e^x = 10$ and $x > 3$.", where e denotes Euler's number, would be such a statement. This new *decidability problem of the real numbers with exponentiation* was solved by Macintyre and Wilkie in 1996 (cf. [2]), but the algorithm they constructed only works under the assumption of *Schanuel's Conjecture* — a mathematical conjecture which has not yet been proved but is often assumed to be correct.

Recent results show that one can construct a similar algorithm if instead of Schanuel's Conjecture one assumes the following:

Every o-minimal exponential field satisfies exactly the same statements as the real exponential field. (1)

In this research project, we investigate under which conditions statement (1) is true.

Terminology

An *ordered field* is a mathematical construct which has similar properties as the real numbers. For example, in an ordered field one can use addition and multiplication and one can compare any two numbers and say which one is larger than the other. General ordered fields, however, can also contain numbers which are infinitely large or infinitely close to 0 (called *infinitesimals*). On some ordered fields one can define an *exponential function* $\exp$, which has the properties "$\exp(x)$ is always positive", $\exp(0) = 1$ and $\exp(x + y) = \exp(x) \cdot \exp(y)$. Again, these are also properties of the standard exponential function e^x or of any other exponentiation on the real numbers (see Figure 1 for examples). An ordered field together with an exponential function is called *ordered exponential field*. The condition of *o-minimality* is more complex to describe. For our purposes, it ensures that functions we define on the ordered exponential field have similar properties to the corresponding functions on the real numbers.

To summarise, o-minimal exponential fields are constructs which, in many ways, behave like the real numbers with its exponential functions. However, they require special care when it comes to the infinitely large and infitesimal elements, since they have no direct correspondence in the real numbers.

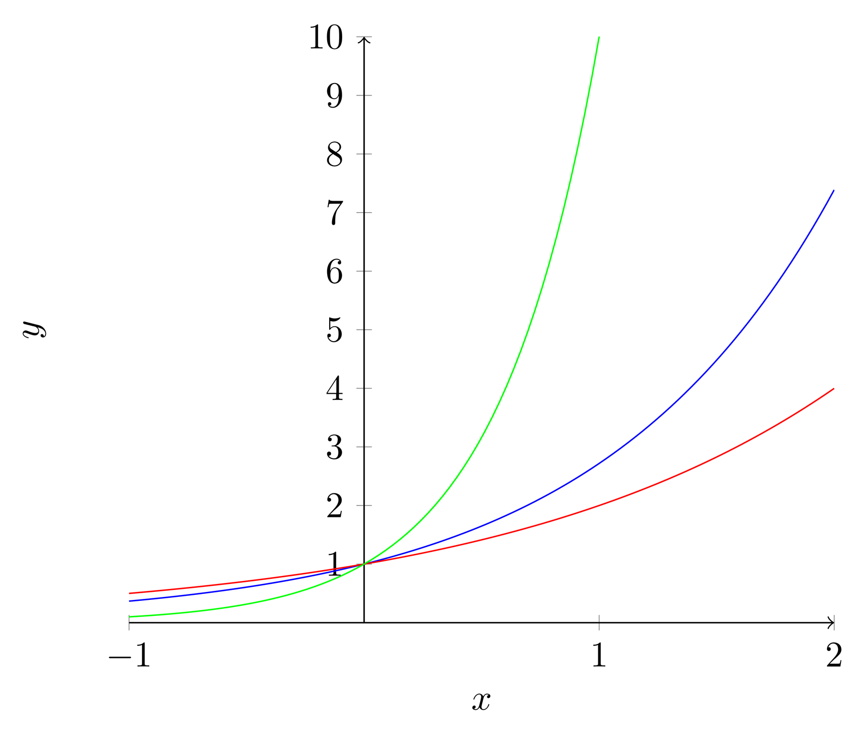

Figure 1: Examples of exponential functions on the real numbers.
2^x, e^x, 10^x

Mathematical Methods

We use methods from the mathematical disciplines model theory and algebra. One of the main ideas of model theory is to examine a certain mathematical structures by looking at similar structures (called *models*) and deducing results in the original structure. This heavily relies on mathematical logic, the foundation of mathematics. Once we look at a certain o-minimal exponential field, algebraic geometry comes into play. This discipline uses abstract algebraic techniques for solving geometrical problems. Geometrical problems in this context are questions about the behaviour of certain functions.

Many properties of exponential function on ordered fields which have infinitely large and infinitesimal elements are treated in [1]. For example, we are interested in the exponential functions mapping a field K to $K^{<0}$, the positive numbers in K, as shown in Figure 2. The condition of o-minimality adds additional properties we can use to refine and extend some of the results in [1].

O-minimal exponential fields with infinitely large and infinitesimal elements can always be associated to some ordered exponential field without infinitely large and infinitesimal elements. This associated exponential field is always contained in the real numbers. Since the real numbers are a widely investigated mathematical construct, we can use properties of exponential functions on the real numbers to draw conclusions in the original exponential field.

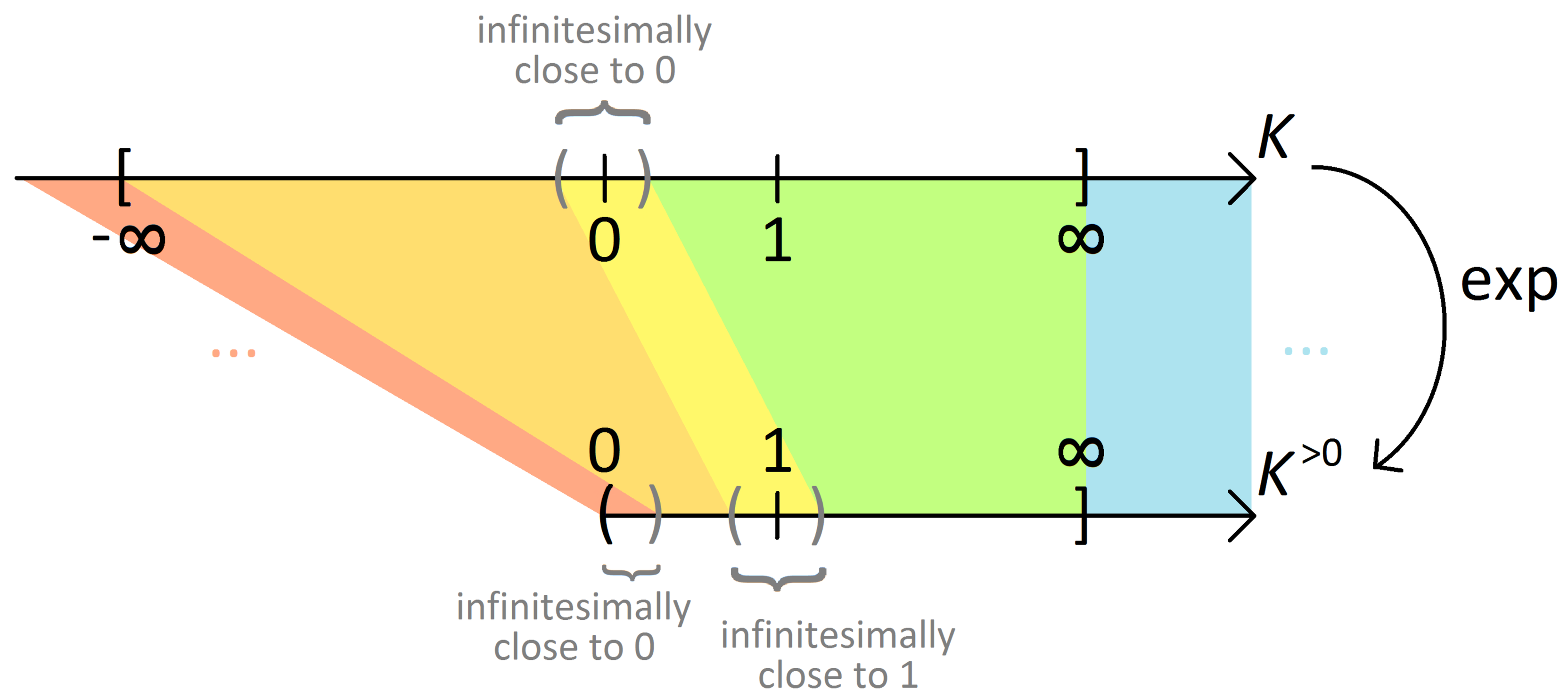

Figure 2: Exponential function on a field with infinitely large and infinitesimal elements.

Recent Results

We have classified all possible exponential functions on fields which are contained in the real numbers. This classification is useful since any exponential field has an associated field in the real numbers.

We have constructed examples that show that o-minimality is a necessary condition in our assertion (1), i. e. that any weaker condition will not give us the required results. From the standpoint of mathematical logic, constructing such examples is an important step, as one always tries to state results as general as possible.

Current Research

Currently we investigate under which conditions o-minimal exponential fields contain certain smaller ordered exponential fields. These smaller ordered exponential fields can be analysed more easily, since we can control the number of elements they contain.

Applications

As a topic from mathematical logic, this work delivers foundational research in pure mathematics. Currently it is analysed how the original decision algorithm for the real numbers without exponentiation can be improved to decrease its complexity. The problem is that its implementation in applied research is still out of reach, as an unrealisticly high amount of computing power would be needed to apply this algorithm to prove or disprove open statements. The decision algorithm for the real numbers with exponentiation is even more complex. Hence, at the current state of research, this project mainly has applications in a theoretical context. However, a future analysis of this decision algorithm may result in its improvement and lead to an actual application to statements about the real numbers.

Main Objectives

1. Finding conditions under which an algorithm can be found that determines whether a given statement about the real numbers with exponentiation is true or false.
2. Examining the connection between Schanuel's Conjecture and conditions for a decidability algorithm of the real exponential field.
3. Investigating the relation of o-minimal exponential fields with their associated exponential field in the real numbers.
4. Analysing properties of exponential functions on o-minimal fields with infinitely large and infinitesimal elements.

References

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[3] A. Tarski, *A decision method for elementary algebra and geometry*, 1948, revised 1951, 2nd ed., RAND Corporation, Santa Monica, 1957.

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