Properties of O-minimal Exponential Fields

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  e.g. $\exists y \; x > \exp(y)$ or $\forall x \exists y \; x > \exp(y)$
- notion of satisfiability

  e.g. $(\mathbb{R}, +, \cdot, 0, 1, <, \exp) \models \neg \forall x \exists y \; x > \exp(y)$

Ordered Exponential Fields

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Definition

Let $(K, +, \cdot, 0, 1, <)$ be an ordered field. A unary function exp which is an order-preserving isomorphism from $(K, +, 0, <)$ to $(K^{>0}, \cdot, 1, <)$ is called an *exponential* on $(K, +, \cdot, 0, 1, <)$. The $\mathcal{L}_{\exp}$-structure $(K, +, \cdot, 0, 1, <, \exp)$ is called an *ordered exponential field*.

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Most prominent example: $\mathbb{R}_{\exp} = (\mathbb{R}, +, \cdot, 0, 1, <, \exp)$ — the real exponential field.

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Example: The formula $\exists y\ x^2 > \exp(y) + \pi$ parametrically defines the set $\{x \in \mathbb{R} \mid \exists y\ x^2 > \exp(y) + \pi\} = (-\infty, -\sqrt{\pi}) \cup (\sqrt{\pi}, \infty)$ over $\mathbb{R}_{\exp}$.

Decidability

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Theorem

An $\mathcal{L}$-structure $\mathcal{M}$ is decidable if and only if $\mathrm{Th}(\mathcal{M})$ is recursively axiomatizable.

Schanuel's Conjecture

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Let $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ be linearly independent over $\mathbb{Q}$. Then

$$\mathrm{td}_{\mathbb{Q}}(\mathbb{Q}(\alpha_1, \ldots, \alpha_n, \mathrm{e}^{\alpha_1}, \ldots, \mathrm{e}^{\alpha_n})) \geq n.$$

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Real Schanuel's Conjecture (SC)

Let $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ be linearly independent over $\mathbb{Q}$. Then

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$\rightarrow$ Macintyre and Wilkie construct a recursive subtheory of $\text{Th}(\mathbb{R}_{\text{exp}})$ which, under the assumption of (SC), is complete.

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Open question: Does (SC) imply (TC)?
This question motivates the study of o-minimal EXP-fields.

Natural Valuation

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Let K be an ordered field. We define an equivalence relation on K by

$$a \sim b \text{ if and only if there exists } n \in \mathbb{Z} \text{ such that } |a| < n|b| \text{ and } |b| < n|a|.$$

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Set $v : K \to G \cup \{\infty\}$ by $v(a) = [a]$ for $a \in K \setminus \{0\}$ and $v(0) = \infty$. v is called the *natural valuation on K*.

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Let K be an ordered field. Let $\mathcal{O} = \{x \in K \mid v(x) \geq 0\}$ and $\mathcal{I} = \{x \in K \mid v(x) > 0\}$. Then $\overline{K} = \mathcal{O}/\mathcal{I}$ defines an archimedean field. This is called the *residue field* of K.

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Proposition

Let $\mathcal{K}_{\exp}$ be an o-minimal EXP-field. Then $\overline{\exp} : \overline{K} \to \overline{K}^{>0}, \overline{a} \mapsto \overline{\exp(a)}$ defines an exponential on $\overline{K}$. Moreover, $\overline{\mathcal{K}}_{\overline{\exp}} = (\overline{K}, +, \cdot, \overline{0}, \overline{1}, <, \overline{\exp})$ is an archimedean EXP-field.

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We call $\overline{\mathcal{K}}_{\overline{\exp}}$ the *residue exponential field* of $\mathcal{K}_{\exp}$.

Archimedean O-minimal Exponential Fields

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Theorem (Laskowski, Steinhorn 1995)

Let $\mathcal{K}_{\mathrm{exp}}$ be an archimedean o-minimal EXP-field. Then $\mathcal{K}_{\mathrm{exp}} \preceq \mathbb{R}_{\mathrm{exp}}$.

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Approach towards (TC): Show that any o-minimal EXP-field has an archimedean prime model.

Approach through Residue Exponential Field

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Let $\mathcal{K}_{\exp}$ be an exponential field such that $\mathcal{K}_{\exp} \equiv \mathbb{R}_{\exp}$. Then $\overline{\mathcal{K}}_{\overline{\exp}} \preceq \mathcal{K}_{\exp}$.

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Assertion

Let $\mathcal{K}_{\exp}$ be an o-minimal EXP-field. Then (SC) implies $\overline{\mathcal{K}}_{\overline{\exp}} \preceq \mathcal{K}_{\exp}$.

Embeddability of the Residue Field

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Theorem

Let $\mathcal{K}_{\mathrm{exp}}$ be a sufficiently saturated o-minimal EXP-field. Then $\overline{\mathcal{K}}_{\overline{\mathrm{exp}}} = \mathbb{R}_{\mathrm{exp}}$. Moreover, assuming (SC), we have $\mathbb{R}_{\mathrm{exp}} \subseteq \mathcal{K}_{\mathrm{exp}}$.

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Corollary

Assume (SC). Let $\mathcal{K}_{\exp}$ be an o-minimal EXP-field. Then $\mathcal{K}_{\exp} \models \mathrm{Th}_{\exists}(\mathbb{R}_{\exp})$.

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Corollary

Assume (SC). Let $\mathcal{K}_{\exp}$ be an o-minimal EXP-field. Then $\mathcal{K}_{\exp} \models \mathrm{Th}_{\exists}(\mathbb{R}_{\exp})$.

Theorem

Let $\mathcal{K}_{\exp}$ be an o-minimal EXP-field. Then the following are equivalent:

1. $\mathrm{Th}_{\exists}(\mathbb{R}_{\exp}) = \mathrm{Th}_{\exists}(\mathcal{K}_{\exp})$ and $\mathcal{K}_{\exp}$ is model complete.
2. (TC): $\mathcal{K}_{\exp} \equiv \mathbb{R}_{\exp}$.

T-convexity

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Theorem

Let $\mathcal{K}_{\exp}$ be an o-minimal EXP-field and $T = \mathrm{Th}(\mathcal{K}_{\exp})$. Then the following are equivalent:

1. $\mathcal{O}$ is T-convex. (i.e. for any continuous 0-definable function f on K we have $f(\mathcal{O}) \subseteq \mathcal{O}$).
2. Any continuous 0-definable function $f : K \to K$ is exponentially bounded (i.e. for any continuous 0-definable function f on K and sufficiently large x we have $|f(x)| < \exp(\exp(\ldots\exp(x)))$) and bounded by an integer on $[0, 1]$.
3. (TC): $\mathcal{K}_{\exp} \equiv \mathbb{R}_{\exp}$.

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3. (TC): $\mathcal{K}_{\text{exp}} \equiv \mathbb{R}_{\text{exp}}$.

This draws a connection between (SC) and the open question whether an o-minimal structure which is not exponentially bounded exists.

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