# Algebraic and Model Theoretic Properties of O-minimal Exponential Fields


## Lothar Sebastian Krapp

Research Group in Real Algebraic Geometry, Fachbereich Mathematik und Statistik, Universität Konstanz, Germany

sebastian.krapp@uni-konstanz.de — +49 7531 88 2788



**Abstract**

The study of the properties of o-minimal exponential fields is motivated by the following open question: Is any o-minimal exponential field whose exponential satisfies the differential equation $\exp' = \exp$ elementarily equivalent to the real exponential field (i.e. the ordered field of real numbers with standard exponentiation). This poster will outline possible approaches and explain connections to other open questions like the decidability of the real exponential field and Schanuel's Conjecture.


## Preliminaries

Language: $\mathcal{L}_{\exp} = (+, \cdot, 0, 1, <, \exp)$, where $\exp$ is a unary function symbol.

**Definition 1.** Let $(K, +, \cdot, 0, 1, <)$ be an ordered field. A unary function $\exp$ which is an order-preserving isomorphism from $(K, +, 0, <)$ to $(K^{>0}, \cdot, 1, <)$ is called an *exponential* on $(K, +, \cdot, 0, 1, <)$. The $\mathcal{L}_{\exp}$-structure $(K, +, \cdot, 0, 1, <, \exp)$ is called an *ordered exponential field*.

Most prominent example: $\mathbb{R}_{\exp} = (\mathbb{R}, +, \cdot, 0, 1, <, \exp)$ — the real exponential field — with complete theory $\mathrm{Th}(\mathbb{R}_{\exp})$.

**Definition 2.** An ordered structure $(M, <, \ldots)$ is called *o-minimal* if every parametrically definable subset of $M$ is a finite union of points and open intervals in $M$.

**Theorem 3** (Wilkie, 1996)**.** *The real exponential field $\mathbb{R}_{\exp}$ is o-minimal.*

## Decidability of $\mathbb{R}_{\exp}$

**Definition 4.** An $\mathcal{L}$-structure $\mathcal{M}$ is called *decidable* if there exists an algorithm that determines whether for a given $\mathcal{L}$-sentence $\varphi$ we have $\mathcal{M} \models \varphi$ or $\mathcal{M} \models \neg\varphi$.

**Theorem 5.** *An $\mathcal{L}$-structure $\mathcal{M}$ is decidable if and only if $\mathrm{Th}(\mathcal{M})$ is recursively axiomatizable.*

**Real Schanuel's Conjecture (SC).**

*Let $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ be linearly independent over $\mathbb{Q}$. Then*

$$\mathrm{td}_{\mathbb{Q}}(\mathbb{Q}(\alpha_1, \ldots, \alpha_n, \mathrm{e}^{\alpha_1}, \ldots, \mathrm{e}^{\alpha_n})) \geq n.$$

Macintyre and Wilkie construct a recursive subtheory of $\mathrm{Th}(\mathbb{R}_{\exp})$ which, under the assumption of (SC), is complete.

**Theorem 6** (Macintyre, Wilkie 1996)**.** *Assume (SC). Then $\mathbb{R}_{\exp}$ is decidable.*

Let $\mathrm{EXP}$ be the $\mathcal{L}_{\exp}$-sentence stating that the differential equation $\exp' = \exp$ holds. We call an ordered exponential field which satisfies $\mathrm{EXP}$ an $\mathrm{EXP}$-field.

**Transfer Conjecture (TC)** *Let $\mathcal{K}_{\exp}$ be an o-minimal $\mathrm{EXP}$-field. Then $\mathcal{K}_{\exp} \equiv \mathbb{R}_{\exp}$.*

**Theorem 7** (Berarducci, Servi 2004)**.** *Assume (TC). Then $\mathbb{R}_{\exp}$ is decidable.*

## Main research question: Does (SC) imply (TC)?

Theorem 7 motivates the study of o-minimal exponential fields.

## Residue Exponential Fields

Let $K$ be an ordered field. We define an equivalence relation on $K$ by

$a \sim b$ if and only if there exists $n \in \mathbb{Z}$ such that $|a| < n|b|$ and $|b| < n|a|$.

The equivalence class of a given $a \in K$ is called the *archimedean equivalence class of* $a$.

Let $G = \{[a] \mid a \in K \setminus \{0\}\}$ and define on $G$ addition by $[a] + [b] = [ab]$ and an order by $[a] < [b]$ if and only if $|a| > |b|$ and $a \not\sim b$. Then $(G, +, <)$ is an ordered group with neutral element $0 = [1]$. It is called the *valuation group of* $K$ under the natural valuation.

Set $v : K \to G \cup \{\infty\}$ by $v(a) = [a]$ for $a \in K \setminus \{0\}$ and $v(0) = \infty$. $v$ is called the *natural valuation on* $K$.

**Definition 8.** Let $K$ be an ordered field. Let $\mathcal{O} = \{x \in K \mid v(x) \geq 0\}$ and $\mathcal{I} = \{x \in K \mid v(x) > 0\}$. Then $\overline{K} = \mathcal{O}/\mathcal{I}$ defines an archimedean field. This is called the *residue field* of $K$.

**Proposition 9.** *Let $\mathcal{K}_{\exp}$ be an o-minimal $\mathrm{EXP}$-field. Then $\overline{\exp} : \overline{K} \to \overline{K}^{>0}, \overline{a} \mapsto \overline{\exp(a)}$ defines an exponential on $\overline{K}$. Moreover, $\overline{\mathcal{K}_{\exp}} = (\overline{K}, +, \cdot, \overline{0}, \overline{1}, <, \overline{\exp})$ is an archimedean $\mathrm{EXP}$-field.*

We call $\overline{\mathcal{K}_{\exp}}$ the *residue exponential field* of $\mathcal{K}_{\exp}$.

**Theorem 10** (Laskowski, Steinhorn 1995)**.** *Let $\mathcal{K}_{\exp}$ be an archimedean o-minimal $\mathrm{EXP}$-field. Then $\mathcal{K}_{\exp} \preceq \mathbb{R}_{\exp}$.*

Approach towards (TC): Show that any o-minimal $\mathrm{EXP}$-field has an archimedean prime model.

## Results

**Theorem 11.** *Let $\mathcal{K}_{\exp}$ be an ordered exponential field such that $\mathcal{K}_{\exp} \equiv \mathbb{R}_{\exp}$. Then $\overline{\mathcal{K}_{\exp}} \preceq \mathcal{K}_{\exp}$.*

**Theorem 12.** *Let $\mathcal{K}_{\exp}$ be an o-minimal $\mathrm{EXP}$-field. Then $\overline{\mathcal{K}_{\exp}} \preceq \mathbb{R}_{\exp}$.*

**Theorem 13.** *Let $\mathcal{K}_{\exp}$ be a sufficiently saturated o-minimal $\mathrm{EXP}$-field. Then $\overline{\mathcal{K}_{\exp}} = \mathbb{R}_{\exp}$. Moreover, assuming (SC), we have $\mathbb{R}_{\exp} \subseteq \mathcal{K}_{\exp}$.*

**Corollary 14.** *Assume (SC). Let $\mathcal{K}_{\exp}$ be an o-minimal $\mathrm{EXP}$-field. Then $\mathcal{K}_{\exp} \models \mathrm{Th}_{\exists}(\mathbb{R}_{\exp})$.*

**Theorem 15.** *Let $\mathcal{K}_{\exp}$ be an o-minimal $\mathrm{EXP}$-field. Then the following are equivalent:*

1. *$\mathrm{Th}_{\exists}(\mathbb{R}_{\exp}) = \mathrm{Th}_{\exists}(\mathcal{K}_{\exp})$ and $\mathcal{K}_{\exp}$ is model complete.*
2. *(TC): $\mathcal{K}_{\exp} \equiv \mathbb{R}_{\exp}$.*

**Theorem 16.** *Let $\mathcal{K}_{\exp}$ be an o-minimal $\mathrm{EXP}$-field and $T = \mathrm{Th}(\mathcal{K}_{\exp})$. Then the following are equivalent:*

1. *$\mathcal{O}$ is $T$-convex. (i.e. for any continuous $0$-definable function $f$ on $K$ we have $f(\mathcal{O}) \subseteq \mathcal{O}$).*
2. *(TC): $\mathcal{K}_{\exp} \equiv \mathbb{R}_{\exp}$.*

This draws a connection between (SC) and the open question whether an o-minimal structure which is not exponentially bounded exists.

## References


[1] S. Kuhlmann, *Ordered Exponential Fields*, The Fields Institute Monographs 12 (American Mathematical Society, Providence, RI, 2000).

[2] A. Macintyre and A. Wilkie, 'On the decidability of the real exponential field', *Kreiseliana: about and around Georg Kreisel* (ed. P. Odifreddi; A. K. Peters, Wellesley, MA, 1996) 441–467.


## Acknowledgements


My research project started in October 2015 and is supervised by Professor Salma Kuhlmann at Universität Konstanz. During my doctoral studies I am financially supported by and a scholar of Carl-Zeiss-Stiftung as well as a scholar of Studienstiftung des deutschen Volkes.