Value Groups and Residue Fields of Models of Real Exponentiation

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Abstract

Let F be an archimedean field, G a divisible ordered abelian group and h a group exponential on G. A triple (F, G, h) is realised in a non-archimedean exponential field $(K, \exp)$ if the residue field of K under the natural valuation is F and the induced exponential group of $(K, \exp)$ is (G, h). We give a full characterisation of all triples (F, G, h) which can be realised in a model of real exponentiation in the following two cases: i) G is countable. ii) G is of cardinality κ and κ-saturated for an uncountable regular cardinal κ with $\kappa^{<\kappa} = \kappa$.

Introduction

The following is well-known: Let F be an archimedean field and G a divisible ordered abelian group. Then the following are equivalent:

1. There exists a non-archimedean $K \models \mathrm{Th}(\mathbb{R}, +, \cdot, 0, 1, <)$ such that the residue group of K under the natural valuation is G and the residue field is F.
2. F is a real closed field and G is divisible.

This shows that in the context of non-archimedean real closed fields, the residue field gives us no information on the value group and vice versa. However, if we expand the structure by an exponential function, then we obtain a richer structure on the value group as well as an interplay between the residue field and the value group.

The aim of this work is to investigate under what conditions an archimedean field together with an abelian group with additional exponential structure can be realised in a model of real exponentiation.

Preliminaries

Let $(K, +, \cdot, 0, 1, <)$ be an ordered field. A unary function $\exp$ which is an order-preserving isomorphism from $(K, +, 0, <)$ to $(K^{>0}, \cdot, 1, <)$ is called an **exponential** on $(K, +, \cdot, 0, 1, <)$. The structure $(K, +, \cdot, 0, 1, <, \exp)$ is called an **ordered exponential field**.

Most prominent example: $\mathbb{R}_{\exp} = (\mathbb{R}, +, \cdot, 0, 1, <, \exp_{\mathbb{R}})$ — the **real exponential field** — with complete theory $T_{\exp}$.

We define an equivalence relation on K by

$a \sim b$ if and only if there exists $n \in \mathbb{Z}$ such that $|a| < n|b|$ and $|b| < n|a|$.

Let $G = \{[a] \mid a \in K \setminus \{0\}\}$ and define on G addition by $[a] + [b] = [ab]$ and an order by $[a] < [b]$ if and only if $|a| > |b|$ and $a \not\sim b$. Then $(G, +, <)$ is an ordered group with neutral element $0 = [1]$. It is called the **valuation group of** K under the natural valuation.

Set $v : K \to G \cup \{\infty\}$ by $v(a) = [a]$ for $a \in K \setminus \{0\}$ and $v(0) = \infty$. v is called the **natural valuation on** K. Similarly we can define a natural valuation v_G on G.

Let $\mathcal{O} = \{x \in K \mid v(x) \geq 0\}$ and $\mathcal{I} = \{x \in K \mid v(x) > 0\}$. Then $\overline{K} = \mathcal{O}/\mathcal{I}$ defines an archimedean field. This is called the **residue field** of K.

There exists an **additive lexicographic decomposition** $K = \mathbf{A} \oplus \mathcal{O}$, where $\mathbf{A}$ is unique up to isomorphism and $v(\mathbf{A}) = G^{<0}$. If $(K^{>0}, \cdot, 1, <)$ is divisible, then there exists a **multiplicative lexicographic decomposition** denoted by $K^{>0} = \mathbf{B} \odot \{x \in K^{>0} \mid v(x) = 0\}$, where $\mathbf{B}$ is order isomorphic to G via $-v$.

Given decompositions of K and $K^{>0}$ as above, if $\exp$ is an exponential on K such that $\exp(\mathbf{A}) = \mathbf{B}$, then the exponential induces an isomorphism $G \to \mathbf{A}$ given by $((-v) \circ \exp)^{-1}$. Applying natural valuations on either side, we obtain an isomorphism $h : v_G(G) \to G^{<0}$, which is called a **group exponential** on G. It is a **strong group exponential** if for any $g \in G^{<0}$ we have $h(v_G(g)) > g$.

A triple (F, G, h), where F is a real closed archimedean field and (G, h) is a divisible exponential group, is called an **exponential valuation triple**.

Main Question

Let (F, G, h) be an exponential valuation triple. What are necessary and sufficient conditions on F, G and h such that there exists a non-archimedean model $\mathcal{K}_{\exp}$ of $T_{\exp}$ realising (F, G, h)?

Results

Residue Fields

Theorem 1. *Let $F \subseteq \mathbb{R}$ be an archimedean field. Then the following are equivalent:*

1. *F is closed under $\exp_{\mathbb{R}}$ and $(F, \exp_{\mathbb{R}}) \preceq \mathbb{R}_{\exp}$.*
2. *There exists a non-archimedean $\mathcal{K}_{\exp} \models T_{\exp}$ with $\overline{K} = F$.*

Countable Groups

Let $\coprod_{\mathbb{Q}} F$ denote the **Hahn sum** of $(F, +, 0, <)$ over $\mathbb{Q}$, i.e. the set $\{s : \mathbb{Q} \to F \mid s(q) = 0 \text{ for all but finitely many elements}\}$ together with pointwise addition and lexicographic order.

Theorem 2. *Let (F, G, h) exponential valuation triple such that G is countable. Then the following are equivalent:*

1. *F is countable, $(F, \exp) \preceq \mathbb{R}_{\exp}$, $G \cong \coprod_{\mathbb{Q}} F$ and h is a strong group exponential.*
2. *There exists a model $\mathcal{K}_{\exp}$ of $T_{\exp}$ realising (F, G, h).*

Theorem 3. *Let μ be an infinite cardinal. Then there exists a non-archimedean $\mathcal{K}_{\exp} \models T_{\exp}$ of cardinality μ such that $v(K)$ is countable if and only if $\mu \leq 2^{\aleph_0}$.*

κ-saturated Groups

Theorem 4. *Let $\kappa = \aleph_\alpha$ be an uncountable regular cardinal with $\sup_{\delta<\alpha} 2^{\aleph_\delta} \leq \aleph_\alpha$. Let (F, G, h) be an exponential valuation triple such that G is κ-saturated and of cardinality κ. Then the following are equivalent:*

1. *$F = \mathbb{R}$ and h is a strong group exponential.*
2. *There exists a model $\mathcal{K}_{\exp}$ of $T_{\exp}$ realising (F, G, h).*

Contraction Groups

A given group exponential h induces a **contraction map** χ on G given by $\chi(g) = \mathrm{sgn}(x) \cdot (h \circ v_G)(x)$ for $x \neq 0$ and $\chi(0) = 0$.

A contraction map is **centripetal** if $x \neq 0$ implies $|x| < |\chi(x)|$. A strong group exponential induces a centripetal contraction map and vice versa (cf. [2]). Hence, in the cases above one can also characterise all triples (F, G, χ), where χ is a contraction map on G rather than a group exponential, by replacing the condition that h is a strong group exponential by the condition that χ is a centripetal contraction.

References

[1] L. S. Krapp, 'Value Groups and Residue Fields of Models of Real Exponentiation', Preprint, 2018, arXiv:1803.03153.

[2] S. Kuhlmann, 'Ordered Exponential Fields', *Fields Inst. Monogr.* 12 (Amer. Math. Soc., Providence, RI, 2000), doi:10.1090/fim/012.

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