Algebraic and Model Theoretic Properties of O-minimal Exponential Fields

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Vorstellung wesentlicher Ergebnisse

1. Motivation

2. Overview of Main Results

3. Residue Fields of Models of Real Exponentiation

Model Theoretic Setting

- fixed language of ordered exponential fields $\mathcal{L}_{\exp} = (+, \cdot, 0, 1, <, \exp)$
- $\mathcal{L}_{\exp}$-structures

  e.g. $(\mathbb{R}, +, \cdot, 0, 1, <, \exp)$
- $\mathcal{L}_{\exp}$-formulas and $\mathcal{L}_{\exp}$-sentences

  e.g. $\exists y \ \ \exp(y) < x$ or $\forall x \exists y \ \ \exp(y) < x$
- notion of satisfiability

  e.g. $(\mathbb{R}, +, \cdot, 0, 1, <, \exp) \models \neg \forall x \exists y \ x > \exp(y)$

Ordered Exponential Fields

Definition

Let $(K, +, \cdot, 0, 1, <)$ be an ordered field. A unary function exp which is an order-preserving isomorphism from $(K, +, 0, <)$ to $(K^{>0}, \cdot, 1, <)$ is called an **exponential** on $(K, +, \cdot, 0, 1, <)$. The $\mathcal{L}_{\exp}$-structure $(K, +, \cdot, 0, 1, <, \exp)$ is called an **ordered exponential field**.

Most prominent example: $\mathbb{R}_{\exp} = (\mathbb{R}, +, \cdot, 0, 1, <, \exp)$ — the **real exponential field**.

O-minimality

Definition

An ordered structure $(M, <, \ldots)$ is called **o-minimal** if every parametrically definable subset of M is a finite union of points and open intervals in M.

Theorem (Wilkie 1996)

The real exponential field $\mathbb{R}_{\exp}$ is o-minimal.

Example: The formula $\exists y\ x^2 > \exp(y) + \pi$ parametrically defines the set $\{x \in \mathbb{R} \mid \exists y\ x^2 > \exp(y) + \pi\} = (-\infty, -\sqrt{\pi}) \cup (\sqrt{\pi}, \infty)$ over $\mathbb{R}_{\exp}$.

Schanuel's Conjecture

Schanuel's Conjecture

Let $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ be linearly independent over $\mathbb{Q}$. Then

$$\mathrm{td}_{\mathbb{Q}}(\mathbb{Q}(\alpha_1, \dots, \alpha_n, \mathrm{e}^{\alpha_1}, \dots, \mathrm{e}^{\alpha_n})) \geq n.$$

$\rightarrow$ Schanuels Conjecture would, for instance, imply the algebraic independence of e and π.

Real Schanuel's Conjecture (SC)

Let $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ be linearly independent over $\mathbb{Q}$. Then

$$\mathrm{td}_{\mathbb{Q}}(\mathbb{Q}(\alpha_1, \dots, \alpha_n, \mathrm{e}^{\alpha_1}, \dots, \mathrm{e}^{\alpha_n})) \geq n.$$

Decidability of the Real Exponential Field

Definition

An $\mathcal{L}$-structure $\mathcal{M}$ is called **decidable** if there exists an algorithm that determines whether for a given $\mathcal{L}$-sentence φ we have $\mathcal{M} \models \varphi$ or $\mathcal{M} \models \neg\varphi$.

Tarski's Exponential Function Problem: Is $\mathbb{R}_{\text{exp}}$ decidable?

Theorem (Macintyre, Wilkie 1996)

Assume (SC). Then $\mathbb{R}_{\text{exp}}$ is decidable.

Transfer Conjecture

- Elementary equivalence: Two $\mathcal{L}$-structures $\mathcal{M}$ and $\mathcal{N}$ are elementarily equivalent if they satisfy exactly the same $\mathcal{L}$-sentences. We write $\mathcal{M} \equiv \mathcal{N}$.
- EXP: $\mathcal{L}_{\exp}$-sentence stating that the differential equation $\exp' = \exp$ with initial condition $\exp(0) = 1$ holds.

Transfer Conjecture (TC)

Let $(K, \exp)$ be an o-minimal EXP-field. Then $(K, \exp) \equiv \mathbb{R}_{\exp}$.

Theorem (Berarducci, Servi 2004)

Assume (TC). Then $\mathbb{R}_{\exp}$ is decidable.

This motivates the study of o-minimal EXP-fields. More specifically, we want to know which properties of $\mathbb{R}_{\exp}$ hold for any o-minimal EXP-field.

1. Value Groups and Residue Fields of Models of Real Exponentiation

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2. Models of True Arithmetic Are Integer Parts of Models of Real Exponentiation

(with Merlin Carl, submitted)

2. Models of True Arithmetic Are Integer Parts of Models of Real Exponentation

- True arithmetic: $\mathrm{Th}(\mathbb{N}, +, \cdot, 0, 1, <)$ — the collection of all sentences which are true in $(\mathbb{N}, +, \cdot, 0, 1, <)$

  e.g. $\forall x \exists y\ (x = y + y \vee x = y + y + 1)$ or $\forall x\ (x \neq 0 \rightarrow x \geq 1)$
- Model of true arithmetic: a structure $(M, +, \cdot, 0, 1, <)$ which satisfies $\mathrm{Th}(\mathbb{N}, +, \cdot, 0, 1, <)$
- Peano arithmetic: a subset of $\mathrm{Th}(\mathbb{N}, +, \cdot, 0, 1, <)$ which implies several properties of the natural numbers including induction
- Integer part: a discretely ordered subring Z of ordered field K such that for any $a \in K$ there exists a unique $k \in Z$ with $k \leq a < k + 1$

  e.g. $\mathbb{Z}$ is an integer part of $\mathbb{Q}$ and of $\mathbb{R}$

2. Models of True Arithmetic Are Integer Parts of Models of Real Exponentiation

Idea:

- Take a model of Peano arithmetic $(M, +, \cdot, 0, 1, <)$ and construct a real closed field $\mathbb{R}_M$ in a similar way as $\mathbb{R}$ is constructed from $\mathbb{N}$. In particular, $M \cup -M$ is an integer part of $\mathbb{R}_M$.
- Construct an exponential function $\exp_M$ on $\mathbb{R}_M$ in a similar way as e^x is defined on $\mathbb{R}$.

Theorem

Let $(M, +, \cdot, 0, 1, <)$ be a model of Peano arithmetic. Then $(\mathbb{R}_M, \exp_M)$ is an ordered EXP-field. If, moreover, $(\mathbb{R}_M, \exp_M)$ is model complete, then $(\mathbb{R}_M, \exp_M)$ is o-minimal.

Theorem

Let $(M, +, \cdot, 0, 1, <)$ be a model of true arithmetic. Then $(\mathbb{R}_M, \exp_M) \equiv \mathbb{R}_{\exp}$.

3. Strongly NIP Almost Real Closed Fields

(with Salma Kuhlmann and Gabriel Lehéricy, submitted)

3. Strongly NIP Almost Real Closed Fields

- Strongly NIP: well-studied model theoretic property generalising o-minimality
- We would like to understand strongly NIP *ordered exponential fields*.
- **Problem:** Not even stronlgy NIP *ordered fields* are fully understood.

3. Strongly NIP Almost Real Closed Fields

- $\mathcal{L}_{\mathrm{or}}$: language of ordered rings $\{+,-,\cdot,0,1,<\}$
- Almost real closed field: An ordered field K is almost real closed with respect to a henselian valuation v if the residue field Kv is real closed.

Theorem

Let K be an almost real closed field with respect to some henselian valuation v. Then K is strongly NIP if and only if the ordered value group vK is strongly NIP.

Conjecture

Any strongly NIP ordered field is either real closed or admits a non-trivial $\mathcal{L}_{\mathrm{or}}$-definable henselian valuation ring.

4. Ordered Fields Dense in Their Real Closure and Definable Convex Valuations

(with Salma Kuhlmann and Gabriel Lehéricy, submitted)

4. Ordered Fields Dense in Their Real Closure and Definable Convex Valuations

- **Fact:** Any ordered field K which is not dense in its real closure admits a non-trivial $\mathcal{L}_{\mathrm{or}}$-definable convex valuation ring.

→ This result combines the study of definable valuations with the study of ordered fields dense in their real closure.

4. Ordered Fields Dense in Their Real Closure and Definable Convex Valuations

Theorem

Let K be an ordered field and let v be a henselian valuation on K such that at least one of the following conditions holds:

1. The value group vK is discretely ordered.
2. The value group vK is not closed in its divisible hull.
3. The residue field Kv is not closed in its real closure.

Then the valuation ring of v is $\mathcal{L}_{\mathrm{or}}$-definable in K.

Natural Valuation

Let K be an ordered field. We define an equivalence relation on K by

$$a \sim b \text{ if and only if there exists } n \in \mathbb{N}_{>0} \text{ such that } |a| \leq n|b| \text{ and } |b| \leq n|a|.$$

The equivalence class of a given $a \in K$ is called the **archimedean equivalence class of** a.

Let $G = \{[a] \mid a \in K \setminus \{0\}\}$ and define on G addition by $[a] + [b] = [ab]$ and an order by $[a] < [b]$ if and only if $|a| > |b|$ and $a \not\sim b$. Then $(G, +, <)$ is an ordered group with neutral element $0 = [1]$. It is called the **value group of** K under the natural valuation.

Set $v\colon K \to G \cup \{\infty\}$ by $v(a) = [a]$ for $a \in K \setminus \{0\}$ and $v(0) = \infty$. The map v is called the **natural valuation on** K**.**

Residue Exponential Field

Definition

Let K be an ordered field. Let $\mathcal{O} = \{x \in K \mid v(x) \geq 0\}$ and $\mathcal{I} = \{x \in K \mid v(x) > 0\}$. Then $\overline{K} = \mathcal{O}/\mathcal{I}$ defines an archimedean field. This is called the **residue field** of K.

Theorem

Let $(K, \exp)$ be an o-minimal EXP-field. Then

$$\overline{\exp}\colon \overline{K} \to \overline{K}^{>0}, \overline{a} \mapsto \overline{\exp(a)}$$

defines an exponential on $\overline{K}$. Moreover, $(\overline{K}, \overline{\exp}) \equiv \mathbb{R}_{\exp}$.

We call $(\overline{K}, \overline{\exp})$ the **residue exponential field** of $(K, \exp)$.

Models of Real Exponentiation

- Model of real exponentiation: an ordered exponential field $(K, \exp)$ with $(K, \exp) \equiv \mathbb{R}_{\exp}$
- $\exp_{\mathbb{R}}$: the standard exponential on $\mathbb{R}$, i.e. $\exp_{\mathbb{R}}(x) = \mathrm{e}^x$.

Theorem (Characterisation of Residue Fields of Models of Real Exponentiation)

Let $F \subseteq \mathbb{R}$ be an archimedean field. Then the following are equivalent:

1. F is closed under $\exp_{\mathbb{R}}$ and $(F, \exp_{\mathbb{R}}) \equiv \mathbb{R}_{\exp}$.
2. There exists a non-archimedean model of real exponentiation $(K, \exp)$ with $\overline{K} = F$.

What have we achieved?

1. Take a non-archimedean o-minimal EXP-field $(K_1, \exp_1)$.
   Reminder: We hope that $(K_1, \exp_1)$ is already a model of real exponentiation.
2. The residue exponential field $(\overline{K_1}, \overline{\exp_1})$ is an archimedean model of real exponentiation.
3. Hence, there exists a non-archimedean model of real exponentiation $(K_2, \exp_2)$ with $\overline{K_2} = \overline{K_1}$.

$\longrightarrow$ *For every non-archimedean o-minimal EXP-field, there exists a non-archimedean model of real exponentiation with same residue exponential field.*

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