Ordered fields dense in their real closure and definable convex valuations

Universität Konstanz

Lothar Sebastian Krapp

joint work with

Salma Kuhlmann and Gabriel Lehéricy

Abstract We undertake a systematic model and valuation theoretic study of the class of ordered fields which are dense in their real closure. We apply this study to determine definable henselian valuations on ordered fields, in the language of ordered rings. In light of our results, we re-examine recent conjectures in the context of strongly NIP ordered fields.

Preliminaries

$\mathcal{L}_{\mathrm{og}} = \{+, 0, <\}$ — **language of ordered groups**
$\mathcal{L}_{\mathrm{or}} = \{+, -, \cdot, 0, 1, <\}$ — **language of ordered rings**

We abbreviate the $\mathcal{L}_{\mathrm{or}}$-structure of an ordered field $(K, +, -, \cdot, 0, 1, <)$ by K and the $\mathcal{L}_{\mathrm{og}}$-structure of an ordered group $(G, +, 0, <)$ by G. All topological notions are meant with respect to the order topology. We say that a valuation v is $\mathcal{L}_{\mathrm{or}}$-definable in an ordered field K if its valuation ring is an $\mathcal{L}_{\mathrm{or}}$-definable subset of K.

Fact. [1, Proposition 6.5] *Let K be an ordered field. Then at least one of the following holds:*
(1) K is dense in its real closure.
(2) K admits a non-trivial $\mathcal{L}_{\mathrm{or}}$-definable convex valuation.

Density in definable closure

For an ordered abelian group G, we denote its divisible hull by G^{div}. For an ordered field K, we denote its real closure by K^{rc}.

Theorem. *Let $\mathcal{L}$ be a language expanding $\mathcal{L}_{\mathrm{og}}$, let $T \supseteq T_{\mathrm{doag}}$ be a complete $\mathcal{L}$-theory which admits quantifier elimination and let $\Sigma \subseteq T$ be a theory extending the theory of ordered abelian groups. Then there is a theory $\Sigma' \supseteq \Sigma$ such that for any $\mathcal{M}' \models T$ and any $\mathcal{M} \subseteq \mathcal{M}'$ with $\mathrm{dcl}(M; \mathcal{M}') = M'$ we have that $\mathcal{M} \models \Sigma'$ if and only if $\mathcal{M} \models \Sigma$ and M is dense in M'. Moreover, if Σ and T are recursive, then we can also choose Σ' to be recursive.*

Corollary. *(1) The class of ordered abelian groups which are dense in their divisible hull is recursively axiomatisable.*
(2) The class of ordered fields which are dense in their real closure is recursively axiomatisable.

Proposition. *Let G be a densely ordered abelian group. Then the following are equivalent:*
(1) G is dense in G^{div}.
(2) G has an archimedean model.
(3) G does not admit a proper non-trivial $\mathcal{L}_{\mathrm{og}}$-definable convex subgroup.

Proposition. *(1) Any ordered field K which has an archimedean model is dense in K^{rc}.*
(2) There exists an ordered field K which is dense in K^{rc} but has no archimedean model.

Contact

Fachbereich Mathematik und Statistik
Universität Konstanz
78457 Konstanz, Germany
`sebastian.krapp@uni-konstanz.de`
+49 7531 88 2788

Dense transcendence bases

For an ordered field K, we denote its transcendence degree over $\mathbb{Q}$ by $\mathrm{td}(K)$.

Proposition. *Let K be an ordered field with $\mathrm{td}(K) \geq \aleph_0$. Then K admits a dense transcendence basis.*

Corollary. *Let K be an ordered field with $\mathrm{td}(K) \geq \aleph_0$. Then there exists a dense subfield $F \subseteq K$ such that $\mathbb{Q}$ is relatively algebraically closed in F.*

Corollary. *Let K be an ordered field with $\mathrm{td}(K) \geq \aleph_0$. Then K is dense in K^{rc} if and only if K admits a transcendence basis which is dense in K^{rc}.*

$\mathcal{L}_{\mathrm{or}}$-definable henselian valuations

Theorem. *Let K be an ordered field and let v be a henselian valuation on K. Suppose that at least one of the following holds.*
(1) The ordered value group vK is discretely ordered.
(2) The ordered value group vK has a limit point in $vK^{\mathrm{div}} \setminus vK$.
(3) The ordered residue field Kv has a limit point in $Kv^{\mathrm{rc}} \setminus Kv$.
Then v is $\mathcal{L}_{\mathrm{or}}$-definable in K. Moreover, in the cases (1) and (2), v is definable by an $\mathcal{L}_{\mathrm{or}}$-formula with one parameter.

Note that, in particular, condition (2) holds if vK is dense in vK^{div}, and that (3) holds if Kv is dense in Kv^{rc}.

Definable valuations and strongly NIP ordered fields

Conjecture. *Any strongly NIP ordered field is either real closed or admits a non-trivial $\mathcal{L}_{\mathrm{or}}$-definable henselian valuation.*

Theorem. [3, Section 5] *The following are equivalent to the above conjecture:*
(1) Any strongly NIP ordered field K admits a henselian valuation v such that Kv is real closed.
(2) For any strongly NIP ordered field K, the natural valuation on K (i.e. the finest convex valuation on K) is henselian.
(3) For any strongly NIP ordered valued field (K, v), whenever v is convex it is already henselian.

Open problems

1. Is there an ordered field K which is dense in K^{rc} but still admits a non-trivial $\mathcal{L}_{\mathrm{or}}$-definable convex valuation?
2. Is there an ordered henselian valued field (K, v) such that v is definable in the language of ordered rings $\{+, -, \cdot, 0, 1, <\}$ but not in the language of rings $\{+, -, \cdot, 0, 1\}$?
3. Let $K = \mathbb{Q}(t)^{\mathrm{rc}}$ ordered by $t > \mathbb{N}$. Is there a dense subfield $F \subseteq K$ such that $\mathbb{Q}$ is relatively algebraically closed in F?
4. Is any strongly NIP ordered field which is dense in its real closure already real closed? In particular, is any strongly NIP archimedean ordered field real closed?

References

[1] F. Jahnke, P. Simon and E. Walsberg, 'Dp-minimal valued fields', *J. Symb. Log.* 82 (2017) 151–165, doi:10.1017/jsl.2016.15.
[2] L. S. Krapp, S. Kuhlmann and G. Lehéricy, 'Ordered fields dense in their real closure and definable convex valuations', submitted, 2019, MODNET Preprint 1577.
[3] L. S. Krapp, S. Kuhlmann and G. Lehéricy, 'Strongly NIP almost real closed fields', submitted, 2019, MODNET Preprint 1563.

Acknowledgements

This work is part of my doctoral research project, which started in October 2015 and is supervised by Professor Salma Kuhlmann at Universität Konstanz. During this part of my doctoral studies I am financially supported and a scholar of Studienstiftung des deutschen Volkes.