Algebraic and Model Theoretic Properties of O-minimal Exponential Fields

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Report on the significant foundation, contents and results of the thesis

1 Motivation

2 Results

Model Theoretic Setting

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- definable sets

  e.g. the $\mathcal{L}_{\exp}$-formula $\exists y \; \exp(y) = x + \pi$ defines the set $(-\pi, \infty)$ in $\mathbb{R}_{\exp}$

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Tarski proved this theorem by presenting an explicit quantifier elimination algorithm.

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Tarski's Exponential Function Problem: Is $\mathbb{R}_{\mathrm{exp}}$ also decidable?

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Real Schanuel's Conjecture (SC)

Let $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ be linearly independent over $\mathbb{Q}$. Then

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Example: The $\mathcal{L}_{\exp}$-formula $\exists y\ x^2 > \exp(y) + \pi$ defines the set $(-\infty, -\sqrt{\pi}) \cup (\sqrt{\pi}, \infty)$ over $\mathbb{R}_{\exp}$.

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Theorem (Berarducci and Servi, 2004)

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Resulting Questions

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Resulting Questions

Schanuel's Conjecture $\Rightarrow$ Decidability Conjecture $\Leftarrow$ Transfer Conjecture

- What are the connections between Schanuel's Conjecture and the Transfer Conjecture?
- What properties of $\mathbb{R}_{\mathrm{exp}}$ can be generalised to any o-minimal EXP-field?
- What are construction methods for o-minimal EXP-fields?

1. Motivation
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Constructions of O-minimal EXP-fields

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- Starting with certain **countable archimedean fields** F and **countable divisible ordered abelian groups** G (both with additional structure), we construct countable models of real exponentiation $(K, \exp)$ with **residue field** F and **value group** G under the natural valuation.

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- Starting with an **arbitrary o-minimal** EXP**-field** $(K, \exp)$, we construct an exponential exp on the real closed field of **surreal numbers No** with $(K, \exp) \preceq (\mathbf{No}, \exp)$.

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- Starting with certain models M of **Peano Arithmetic**, we construct o-minimal EXP-fields with **integer part** $M \cup (-M)$.

Properties of $\mathbb{R}_{\text{exp}}$ Generalised to O-minimal EXP-fields

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- For any o-minimal EXP-field $(K, \exp)$, we have $(\overline{K}, \overline{\exp}) \preceq \mathbb{R}_{\exp}$. Here, $\overline{K}$ is the residue field of K under the natural valuation and $\overline{\exp}$ is the exponential induced on the residue field.

Connections between Schanuel's Conjecture and Transfer Conjecture

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- Assuming (SC), any o-minimal EXP-field satisfies the existential theory $\mathrm{Th}_{\exists}(\mathbb{R}_{\mathrm{exp}})$ of $\mathbb{R}_{\mathrm{exp}}$.

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- Assuming (SC), any o-minimal EXP-field satisfies the existential theory $\mathrm{Th}_\exists(\mathbb{R}_{\mathrm{exp}})$ of $\mathbb{R}_{\mathrm{exp}}$.
- Assuming (TC), if some o-minimal EXP-field satisfies Schanuel's Conjecture, then all o-minimal EXP-fields do so.

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