Neural Networks, NIP and Definable Valuations

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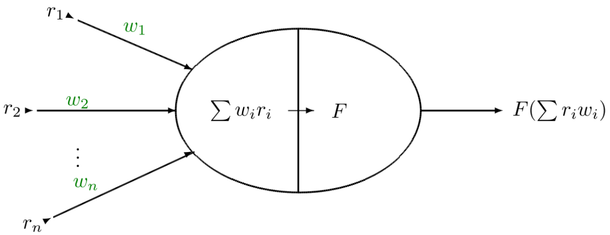
1. Neural Networks
2. NIP
3. Definable Valuations

Artificial Neurons



r_i: real numbers, **input**

w_i: real numbers, **weights**

$\sum w_i r_i$: weighted sum

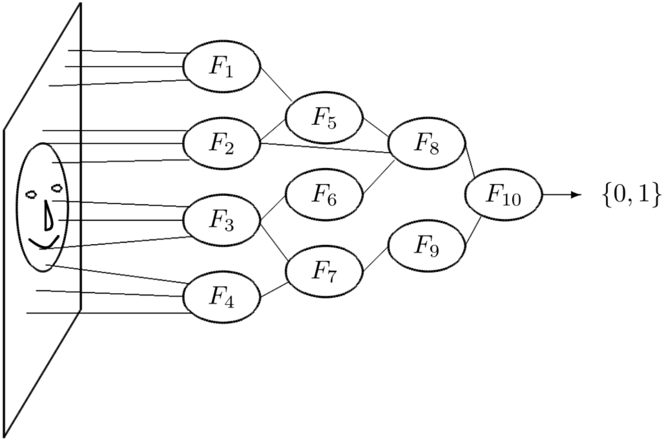
F: real valued function,
activation function

Activation Functions

Typical activation functions F:

- characteristic functions on intervals (a, ∞)
- piecewise linear functions
- sigmoid function $F(t) = \frac{1}{1+\exp(-t)}$

Artificial Neural Network



(10 neurons, 4 layers)

Artificial Neural Network

X: input space, e.g. $(\mathbb{R}^2 \times \{0, \ldots, 255\})^{12}$
Y: output space, e.g. $\{0, 1\}$
$F_i(\underline{x}, \underline{w})$: activation functions

The network computes a class of functions $X \to Y$.

Learning Cycle

1. network is in an initial state h coded by the weights
2. training sample $(x, y) \in X \times Y$ is chosen
3. $h(x)$ is computed
4. the weights are adjusted depending on $h(x) = y$ or $h(x) \neq y$ (also considering previous training samples)

Goal: After finitely many training samples the network is in a state h which gives a good approximation to recognising the pattern.

Formal Learning

Neural network H: set of all possible functions depending on the weights

Sample space $Z = X \times Y$

Learning algorithm L:

$$L : \bigcup_{m=1}^{\infty} Z^m \to H.$$

Learning Algorithm

p – probability measure on Z measuring the probability that a sample is chosen as training sample

$\mathrm{er}_p(h) = p\{(x, y) \in Z \mid h(x) \neq y\}$ – error of $h \in H$

$\mathrm{opt}_p(H) = \inf_{h \in H} \mathrm{er}_p(h)$ – best approximation in H for given p

Learning Algorithm

Definition

Let H be a collection of functions $X \to Y$ for a given sample space $Z = X \times Y$. A **learning algorithm** L is a map

$$L : \bigcup_{m=1}^{\infty} Z^m \to H$$

with the following property:
$\forall \varepsilon, \delta \in (0,1)\ \exists m_0 \in \mathbb{N}\ \forall m \geq m_0 :$
for any probability measure p on Z we have

$$p^m\{z \in Z^m \mid \mathrm{er}_p(L(z)) < \mathrm{opt}_p(H) + \varepsilon\} \geq 1 - \delta,$$

where p^m is the product measure on Z^m.
H is called **learnable** if there exists a learning algorithm for H.

Model Theoretic Setting

- first-order languages $\mathcal{L}$:
  $\mathcal{L}_{\mathrm{r}} = (+, -, \cdot, 0, 1)$, $\mathcal{L}_{\mathrm{or}} = (+, -, \cdot, 0, 1, <)$, $\mathcal{L}_{\exp} = (+, -, \cdot, 0, 1, <, \exp)$
- $\mathcal{L}$-structures:
  $(\mathbb{Z}, +, -, \cdot, 0, 1)$, $(\mathbb{Q}, +, -, \cdot, 0, 1, <)$, $(\mathbb{R}, +, -, \cdot, 0, 1, <, \exp)$
- $\mathcal{L}$-formulas and $\mathcal{L}$-sentences:
  $\forall x \exists y \ x + y = 0$
  $\exists y \ y < x$
  $\forall x \exists y \ \exp(y) < x$
- complete $\mathcal{L}$-theories:
  $\mathrm{Th}(\mathbb{R}, +, -, \cdot, 0, 1, <)$ (the theory of **real closed fields**) – set of all $\mathcal{L}_{\mathrm{or}}$-sentences which are true in $(\mathbb{R}, +, -, \cdot, 0, 1, <)$

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- $\mathcal{L}$-structures:
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- $\mathcal{L}$-formulas and $\mathcal{L}$-sentences:
  $\forall x \exists y \; x + y = 0$ – true in $(\mathbb{R}, +, -, \cdot, 0, 1, <)$
  $\exists y \; y < x$
  $\forall x \exists y \; \exp(y) < x$ – false in $(\mathbb{R}, +, -, \cdot, 0, 1, <, \exp)$
- complete $\mathcal{L}$-theories:
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VC Dimension

Let $\mathcal{L}$ be a first-order language, let $\mathcal{M} = (M, \ldots)$ be an $\mathcal{L}$-structure and let $\varphi(\underline{x}, \underline{y})$ be an $\mathcal{L}$-formula.

Definition

The **Vapnik–Chervonenkis dimension** (**VC dimension**) of φ (in $\mathcal{M}$) is defined as

$$\mathrm{vc}(\varphi(\underline{x}; \underline{y})) := \max(S_\varphi)$$

where $S_\varphi \subseteq \mathbb{N}$ consists of all $n \in \mathbb{N}$ with the following property:

There exist tuples $(\underline{a}_i)_{i<n}$ and $(\underline{b}_J)_{J \subseteq \{0,\ldots,n-1\}}$ in M such that for any $i < n$ and any $J \subseteq \{0, \ldots, n-1\}$

$$\varphi(\underline{a}_i; \underline{b}_J) \text{ holds in } \mathcal{M} \text{ if and only if } i \in J.$$

If S_φ has no maximum, then $\mathrm{vc}(\varphi(\underline{x}; \underline{y})) = \infty$.

VC Dimension

Example

Consider the $\mathcal{L}_{\mathrm{r}}$-structure $(\mathbb{Z}, +, -, \cdot, 0, 1)$. The $\mathcal{L}_{\mathrm{r}}$-formula $\varphi(x; y)$ given by

$$\exists d \; d \cdot x = y$$

expresses "x divides y".

For any $i \in \mathbb{N}$, let $a_i \in \mathbb{Z}$ be the i-th prime number. For any $n \in \mathbb{N}$ and any $J \subseteq \{0, \ldots, n-1\}$, let

$$b_J = \prod_{i \in J} p_i.$$

Then $\varphi(a_i; b_J)$ holds in $(\mathbb{Z}, +, -, \cdot, 0, 1)$ if and only if $i \in J$. Hence, $\mathrm{vc}(\varphi(x; y)) = \infty$.

NIP

Definition

A formula $\varphi(\underline{x};\underline{y})$ has the **independence property** (or is **IP**) if $\mathrm{vc}(\varphi)=\infty$. If φ does not have the independence property, it is called **NIP**.

An $\mathcal{L}$-structure $\mathcal{M}$ is NIP: For **every** $\mathcal{L}$-structure $\mathcal{N}$ satisfying $\mathrm{Th}(\mathcal{M})$, **every** $\mathcal{L}$-formula $\varphi(\underline{x};\underline{y})$ is NIP in $\mathcal{N}$.

Examples of NIP structures:

- o-minimal structures
- weakly o-minimal structures
- C-minimal structures
- ...

O-minimality

Definition

An ordered structure $(M, <, \ldots)$ is called **o-minimal** if every parametrically definable subset of M is a finite union of points and open intervals in M.

Theorem (Wilkie 1996)

The real exponential field $\mathbb{R}_{\text{exp}} = (\mathbb{R}, +, -, \cdot, 0, 1, <, \exp)$ is o-minimal.

Example: The formula $\exists y\ x^2 > \exp(y) + \pi$ parametrically defines the set $\{x \in \mathbb{R} \mid \exists y\ x^2 > \exp(y) + \pi\} = (-\infty, -\sqrt{\pi}) \cup (\sqrt{\pi}, \infty)$ over $\mathbb{R}_{\text{exp}}$.

NIP Implies Learnability

Theorem

Let $\mathcal{R} = (\mathbb{R}, +, \cdot, <, \dots)$ be an NIP expansion of $(\mathbb{R}, +, \cdot, <)$, let $X \subseteq \mathbb{R}^d$ be a (parametrically) definable set over $\mathcal{R}$ and let H be a collection of activation functions of a neural network $X \to \{0, 1\}$ (parametrically) definable over $\mathcal{R}$. Then H is learnable.

Since $\mathbb{R}_{\text{exp}}$ is o-minimal and thus NIP, any set H of $\mathbb{R}_{\text{exp}}$-definable activation functions of a neural network is learnable.

Brief Historical Overview

- **Vapnik, Chervonenkis, 1971:** paper on statistical learning theory introducing the notion of VC dimensions (for sets rather than formulas)
- **Shelah, 1971:** paper on model theory introducing the independence property
- **Pillay, Steinhorn, 1986:** proof that o-minimality implies NIP
- **Laskowski, 1992:** connecting the notion of VC dimension to the independence property
- **Wilkie, 1996:** proof that $\mathbb{R}_{\mathrm{exp}}$ is o-minimal

Classification Question

What are necessary and sufficient conditions on a general ordered field $(K, +, -, \cdot, 0, 1, <)$ to be NIP?

With view to neural network learning: What ordered subfields

$$(K, +, -, \cdot, 0, 1, <) \subseteq (\mathbb{R}, +, -, \cdot, 0, 1, <)$$

are NIP?

(Note that $(\mathbb{Q}, +, -, \cdot, 0, 1, <)$ is **not** NIP!)

Shelah–Hasson Conjecture

Shelah–Hasson Conjecture (specialised to ordered fields)

Any (strongly) NIP ordered field $(K, +, -, \cdot, 0, 1, <)$ is either real closed or contains a non-trivial henselian valuation ring that is definable in the language $\mathcal{L}_{\mathrm{or}}$.

This conjecture would imply that the NIP ordered subfields of $(\mathbb{R}, +, -, \cdot, 0, 1, <)$ are exactly its real closed subfields!

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Convex Valuation Rings

Definition

Let $(K, +, -, \cdot, 0, 1, <)$ be an ordered field. A **convex valuation ring** in K is a convex subring $R \subseteq K$ such that for any $a \in K^\times$

$$a \in R \text{ or } a^{-1} \in R.$$

Maximal ideal (of infinitesimals): $I = \{a \in K^\times \mid a^{-1} \notin R\} \cup \{0\}$.

Residue field: $\overline{K} = R/I$.

Residue elements: For $a \in R$ we write $\overline{a} = a + I$.

Convex Valuation Rings

Examples

- For any ordered subfield $(K, +, -, \cdot, 0, 1, <)$ of $(\mathbb{R}, +, -, \cdot, 0, 1, <)$, the only convex valuation ring in K is $R = K$ (the trivial valuation ring).
- The field of Laurent series $\mathbb{R}(\!(x)\!)$ in one variable over $\mathbb{R}$ can be ordered by setting $\frac{1}{x} > r$ for any $r \in \mathbb{R}^{>0}$. Its only non-trivial convex valuation ring is the power series ring $\mathbb{R}[\![x]\!]$.
- The only $\mathcal{L}_{\mathrm{or}}$-definable convex valuation ring in an o-minimal (or, equivalently, real closed) ordered field is the trivial valuation ring.

Henselian Valuation Rings

Definition

Let $(K, +, -, \cdot, 0, 1, <)$ be an ordered field. A convex valuation ring R in K is **henselian** if for any polynomial

$$p(x) = a_n x^n + \ldots + a_0 \in R[x]$$

and any simple zero $a \in \overline{K}$ of

$$\overline{p}(x) = \overline{a_n} x^n + \ldots + \overline{a_0} \in \overline{K}[x],$$

there exists a zero $b \in R$ of $p(x)$ with $\overline{b} = a$.

Motto: *Simple zeros can be lifted from the residue field.*

Henselian Valuation Rings

Examples

- Any trivial valuation ring is henselian, as the residue field coincides with the field.
- $\mathbb{R}[\![x]\!]$ is a henselian valuation ring of $\mathbb{R}(\!(x)\!)$.

Approach towards Shelah–Hasson Conjecture

Refinements of the property 'NIP':

o-minimal $\rightarrow$ **weakly o-minimal**
$\rightarrow$ **dp-minimal** $\rightarrow$ **dp-finite**
$\rightarrow$ **strongly NIP** $\rightarrow$ **NIP**

Currently the Shelah–Hasson Conjecture specialised to ordered fields has been verified for the dp-minimal case. Hence, an ordered subfield of $(\mathbb{R}, +, -, \cdot, 0, 1, <)$ is dp-minimal if and only if it is real closed.

Strongly NIP Ordered Fields

Theorem (K., Kuhlmann, Lehéricy)

The following are equivalent:

- The Shelah–Hasson Conjecture specialised to ordered fields: Any strongly NIP ordered field is either real closed or admits a non-trivial $\mathcal{L}_{\mathrm{or}}$-definable henselian valuation ring.
- Any strongly NIP ordered field admits a henselian valuation ring with real closed residue field.

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