# Definable Henselian Valuations by Conditions on the Value Group

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# Valuation Rings

**Definition**

Let $(K,+,-,\cdot,0,1)$ be a field. A **valuation ring** in $K$ is a subring $R \subseteq K$ such that for any $a \in K^{\times}$

$$a \in R \text{ or } a^{-1} \in R.$$

**Maximal ideal**: $I = \{a \in K^{\times} \mid a^{-1} \notin R\} \cup \{0\}$.

**Residue field**: $\overline{K} = R/I$.

**Residue elements**: For $a \in R$ we write $\overline{a} = a + I$.

# Valuations

Let $(K, +, -, \cdot, 0, 1)$ be a field and let $R \subseteq K$ be a valuation ring.
Define an equivalence relation on $K^\times$:

$$a \sim_R b \;:\Leftrightarrow \tfrac{a}{b} \in R \wedge \tfrac{b}{a} \in R.$$

Then $G = \{[a] \mid a \in K^\times\}$ forms an ordered abelian group: $[a] + [b] = [ab]$ and $[a] < [b]$ if and only if $\frac{a}{b} \notin R$.
The map

$$v\colon K \to G \cup \{\infty\}, \begin{cases} a \mapsto [a] & \text{if } a \in K^\times, \\ 0 \mapsto \infty & \end{cases}$$

is the **valuation** on $K$ with value group $G$, valuation ring $R = \{a \in K \mid v(a) \geq 0\}$ and valuation ideal $I = \{a \in K \mid v(a) > 0\}$.

# Valuations

**Examples**

- For any field $K$ the trivial valuation $v\colon K^\times \to \{0\}$ has the valuation ring $K$ and the residue field $K$.
- The field of Laurent series $k((x))$ with coefficient field $k$ can be equipped with the valuation

$$v_{\min}\left(\sum_{\ell=m}^{\infty} a_\ell x^\ell\right) = m$$

(where $a_m \neq 0$) with value group $\mathbb{Z}$, valuation ring $k[\![x]\!]$ and residue field $k$.
- For any totally ordered field $(K, +, -, \cdot, 0, 1, <)$, any convex subring of $K$ is a valuation ring of $K$. The finest convex valuation ring on $K$ produces an archimedean residue field.

# Henselian Valuations

## Definition

Let $(K, +, -, \cdot, 0, 1)$ be a field. A valuation $v$ with valuation ring $R$ on $K$ is **henselian** if for any polynomial

$$p(x) = a_n x^n + \ldots + a_0 \in R[x]$$

and any simple zero $a \in \overline{K}$ of

$$\overline{p}(x) = \overline{a_n} x^n + \ldots + \overline{a_0} \in \overline{K}[x],$$

there exists a zero $b \in R$ of $p(x)$ with $\overline{b} = a$.

Motto: *Simple zeros can be lifted from the residue field.*

# Henselian Valuation Rings

**Examples**

- Any trivial valuation is henselian, as the residue field coincides with the field.
- The valuation $v_{\text{min}}$ on $k((x))$ is henselian.
- If $(K, +, -, \cdot, 0, 1, <)$ is a real closed ordered field, any valuation on $K$ with convex valuation ring is henselian.

1. Valuations
2. Definability
3. Conditions on the Value Group

# Model Theoretic Setting

- first-order languages $\mathcal{L}$:
  $\mathcal{L}_{\mathrm{r}} = (+, -, \cdot, 0, 1)$, $\mathcal{L}_{\mathrm{or}} = (+, -, \cdot, 0, 1, <)$
- $\mathcal{L}$-structures:
  $(\mathbb{Z}, +, -, \cdot, 0, 1)$, $(\mathbb{R}, +, -, \cdot, 0, 1, <)$
- $\mathcal{L}$-formulas and $\mathcal{L}$-sentences:
  $\forall x \exists y \; x + y = 0$
- $\mathcal{L}$-definability (*without* and *with* parameters):
  $\exists y \; x \cdot y = 1$ defines $\{1, -1\}$ in $(\mathbb{Z}, +, -, \cdot, 0, 1)$ and $\mathbb{R} \setminus \{0\}$ in $(\mathbb{R}, +, -, \cdot, 0, 1, <)$.
  $x \cdot x < \sqrt{3}$ defines $(-\sqrt[4]{3}, \sqrt[4]{3})$ in $(\mathbb{R}, +, -, \cdot, 0, 1, <)$.

# Main Questions

A valuation $v$ is said to be $\mathcal{L}$-**definable** if its valuation ring $R$ is $\mathcal{L}$-definable.

Let $(K, +, -, \cdot, 0, 1)$ be a field and let $v$ be a henselian valuation on $K$.

**What are necessary and sufficient conditions such that $v$ is $\mathcal{L}_{\mathrm{r}}$-definable?**

# Main Questions

A valuation $v$ is said to be $\mathcal{L}$-**definable** if its valuation ring $R$ is $\mathcal{L}$-definable.

Let $(K, +, -, \cdot, 0, 1, <)$ be an ordered field and let $v$ be a henselian valuation on $K$.

**What are necessary and sufficient conditions such that $v$ is $\mathcal{L}_{\mathrm{or}}$-definable?**

# Motivation: Decidability

**Theorem (Ax, 1965)**

*Let $k$ be a field. Then $v_{\min}$ is (parameter-free) $\mathcal{L}_{\mathrm{r}}$ definable in $k((x))$.*

Since the residue field of $v_{\min}$ is $k$, Ax concludes: If $(k, +, -, \cdot, 0, 1)$ is undecidable, then $(k((x)), +, -, \cdot, 0, 1)$ is undecidable.

# Motivation: Classification of NIP Fields

**What are necessary and sufficient conditions on a field (an ordered field) to be NIP?**

**Examples of NIP structures:**

- o-minimal structures
- weakly o-minimal structures
- C-minimal structures
- ...

# Motivation: Classification of NIP Fields

**Shelah–Hasson Conjecture**

Any infinite NIP field $(K, +, -, \cdot, 0, 1)$ is either algebraically closed, real closed or admits a non-trivial $\mathcal{L}_{\mathrm{r}}$-definable henselian valuation.

Refinements of the property 'NIP':

**dp-minimal** $\rightarrow$ **dp-finite**
$\rightarrow$ **strongly NIP** $\rightarrow$ **NIP**

Currently the Shelah–Hasson Conjecture has been verified for the dp-finite case (Johnson, 2020).

# Main Questions – Revised

Let $(K, +, -, \cdot, 0, 1)$ be a field and let $v$ be a henselian valuation on $K$.

**What are necessary and sufficient conditions on the residue field and the value group such that $v$ is $\mathcal{L}_{\mathrm{r}}$-definable?**

*We concentrate on the value group.*

# Main Questions – Revised

Let $(K, +, -, \cdot, 0, 1, <)$ be an ordered field and let $v$ be a henselian valuation on $K$.

**What are necessary and sufficient conditions on the residue field and the value group such that $v$ is $\mathcal{L}_{\mathrm{or}}$-definable?**

*We concentrate on the value group.*

# Discretely Ordered Case

An ordered abelian group $(G, +, 0, <)$ is **discretely ordered** if there exists a least positive element in $G$.

**Theorem (Hong, 2013)**

*Let $(K, +, -, \cdot, 0, 1)$ be a field and let $v$ be a henselian valuation on $K$. Suppose that its value group $G$ is discretely ordered. Then $v$ is $\mathcal{L}_{\mathrm{r}}$-definable (using one parameter).*

One can generally not obtain definability without parameters, even if one allows the richer language $\mathcal{L}_{\mathrm{or}}$ (K., Kuhlmann, Link, 2021). However, if $(G, +, 0, <)$ is already elementarily equivalent to $(\mathbb{Z}, +, 0, <)$, then $v$ is already $\mathcal{L}_{\mathrm{r}}$-definable *without parameters*.

# Densely Ordered Case

An ordered abelian group $(G, +, 0, <)$ is **densely ordered** if it is not discretely ordered. The **divisible hull** of $G$ is given by $G^{\mathrm{div}} = \{\frac{g}{n} \mid g \in G, n \in \mathbb{N}\}$.

**Theorem (K., Kuhlmann, Link, 2021)**

*Let $(K, +, -, \cdot, 0, 1)$ be a field and let $v$ be a henselian valuation on $K$. Suppose that its value group $G$ is not closed in $G^{\mathrm{div}}$ with respect to the order-topology. Then $v$ is $\mathcal{L}_{\mathrm{r}}$-definable (using one parameter).*

Examples of such ordered abelian groups are densely ordered subgroups of $\mathbb{Q}$. Again, one can generally not obtain definability without parameters, even if one allows the richer language $\mathcal{L}_{\mathrm{or}}$.

# References


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