Topological Properties of Ordered Abelian Groups and Definable Henselian Valuations

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1. Ordered Abelian Groups
2. Henselian Valuations
3. Definability

Definition

Ordered abelian group

Let $(G, +, 0)$ be an *abelian group* and let $<$ be a *strict linear order* on G. Then $(G, +, -, 0, <)$ is an **ordered abelian group** if for any $a, b, c \in G$:

$$a < b \to a + c < b + c.$$

We fix the language of ordered groups $\mathcal{L}_{\mathrm{og}} = (+, -, 0, <)$. If the operations and linear ordering are clear from the context, then we simply write G for an ordered abelian group.

Divisible Hull

Definition

Let G be an ordered abelian group. We define its divisible hull G^{div} by

$$G^{\text{div}} := \left\{ \tfrac{g}{n} \mid g \in G, n \in \mathbb{N}_{>0} \right\}.$$

G^{div} becomes an ordered abelian group with usual addition

$$\frac{g}{n} + \frac{h}{m} = \frac{mg+nh}{nm}$$

and ordering $\frac{g}{n} > 0$ if and only if $g > 0$.
If $G = G^{\text{div}}$, then G is called **divisible**.
Moreover, G is a topological subspace of G^{div} with respect to the order-topology.

Discrete and Dense Orderings

Definition

An ordered abelian group G is **discretely ordered** if there exists a least positive element, i.e. an element $1 \in G$ with $0 < 1$ and $(0, 1) \cap G = \emptyset$. Otherwise, it is called **densely ordered**.

Note that a group G is densely ordered if and only if for any $a, b \in G$ with $a < b$ there exists $c \in G$ such that $a < c < b$.

Archimedean Groups

Hölder

An ordered abelian group G is **archimedean** if and only if it can be embedded (as an $\mathcal{L}_{\mathrm{og}}$-structure) into $(\mathbb{R}, +, -, 0, <)$.

Hence, the divisible hull of an archimedean ordered abelian group is also archimedean.

Dichotomy: An archimedean ordered abelian group is either isomorphic to $(\mathbb{Z}, +, -, 0, <)$ (in which case it is **closed in its divisible hull**) or **dense in its divisible hull**.

What about the non-archimedean case?

Examples of Non-Archimedean Groups

Let G and H be non-trivial ordered abelian groups. Its **lexicographic sum** $G \oplus H$ is an ordered abelian group with the linear ordering

$$(g,h) < (g',h') :\Leftrightarrow [g < g' \vee (g = g' \wedge h < h')].$$

Any lexicographic sum is non-archimedean.

- $\mathbb{Q} \oplus \mathbb{Q}$ is densely ordered and divisible, thus both dense and closed in its divisible hull $\mathbb{Q} \oplus \mathbb{Q}$.
- $\mathbb{Z} \oplus \mathbb{Z}$ and $\mathbb{Q} \oplus \mathbb{Z}$ are discretely ordered, thus closed in their divisible hull $\mathbb{Q} \oplus \mathbb{Q}$.
- $\mathbb{Z} \oplus \mathbb{Q}$ is densely ordered but closed in its divisible hull $\mathbb{Q} \oplus \mathbb{Q}$!
  E.g. $\mathbb{Z} \oplus \mathbb{Q}$ contains no element strictly between $(\frac{1}{2}, -1)$ and $(\frac{1}{2}, 1)$, so $(\frac{1}{2}, 0) \in \mathbb{Q} \oplus \mathbb{Q}$ is no limit point of $\mathbb{Z} \oplus \mathbb{Q}$.

Densely Ordered

- A lexicographic sum $G \oplus H$ is densely ordered if and only if H is densely ordered.

  In particular, if H is archimedean, then $G \oplus H$ is discretely ordered if and only if H is isomorphic to $\mathbb{Z}$.

- A **Hahn sum** $\bigoplus_{i \in \mathbb{N}} G_i = G_0 \oplus G_1 \oplus \ldots$ is *always* densely ordered.

  In particular, $\mathbb{Z} \oplus \mathbb{Z} \oplus \ldots$ is densely ordered.

Closed in Divisible Hull

- Every discretely ordered abelian group is closed in its divisible hull.
- If G is non-divisible and H is divisible, then $G \oplus H$ is densely ordered and closed in its divisible hull. (E.g. $\mathbb{Z} \oplus \mathbb{Q}$.)
- A Hahn sum $\bigoplus_{i \in \mathbb{N}} G_i = G_0 \oplus G_1 \oplus \ldots$ is either divisible (i.e. each component G_i is divisible) or closed in its divisible hull.

Dense in Divisible Hull

Theorem (K., Kuhlmann, Lehéricy; and others)

Let G be a densely ordered abelian group. Then the following are equivalent:

1. G is dense in its divisible hull.
2. G is $\mathcal{L}_{\mathrm{og}}$-elementarily equivalent to some archimedean ordered abelian group.
3. G is regular: for any $n \in \mathbb{N}_{>0}$ and any infinite convex subset I of G there is some n-divisible element in I.
4. G has no proper non-trivial $\mathcal{L}_{\mathrm{og}}$-definable convex subgroup.

Further Examples

Let $A = \{\frac{a}{2^n} \mid a \in \mathbb{Z}, n \in \mathbb{N}\}$ (the 2-divisible hull of $\mathbb{Z}$). Consider the following non-divisible densely ordered abelian groups with divisible hull $\mathbb{Q} \oplus \mathbb{Q}$:

- $\mathbb{Q} \oplus A$ is dense in its divisible hull.
  E.g. $(0, \frac{1}{3}) < (0, \frac{1}{2}) < (0, \frac{2}{3})$.
- $A \oplus \mathbb{Q}$ is closed in its divisible hull.
  E.g. $A \oplus \mathbb{Q}$ contains no element strictly between $(\frac{1}{3}, -1)$ and $(\frac{1}{3}, 1)$, so $(\frac{1}{3}, 0)$ is no limit point of $A \oplus \mathbb{Q}$.
- $A \oplus A$ is neither dense nor closed in its divisible hull.
  E.g. again $(\frac{1}{3}, 0) \in \mathbb{Q} \oplus \mathbb{Q}$ is no limit point of $A \oplus A$ but $(0, \frac{1}{3}) \notin A \oplus A$ is a limit point of $A \oplus A$.

Valuation Rings

Definition

Let $(K, +, -, \cdot, 0, 1)$ be a field. A **valuation ring** in K is a subring $R \subseteq K$ such that for any $a \in K^\times$

$$a \in R \text{ or } a^{-1} \in R.$$

Maximal ideal: $I = \{a \in K^\times \mid a^{-1} \notin R\} \cup \{0\}$.

Residue field: $\overline{K} = R/I$.

Residue elements: For $a \in R$ we write $\overline{a} = a + I$.

Valuations

Let $(K, +, -, \cdot, 0, 1)$ be a field and let $R \subseteq K$ be a valuation ring.
Define an equivalence relation on $K^\times$:

$$a \sim_R b \; :\Leftrightarrow \tfrac{a}{b} \in R \wedge \tfrac{b}{a} \in R.$$

Then $G = \{[a] \mid a \in K^\times\}$ forms an ordered abelian group: $[a] + [b] = [ab]$ and $[a] < [b]$ if and only if $\frac{a}{b} \notin R$.
The map

$$v \colon K \to G \cup \{\infty\}, \begin{cases} a \mapsto [a] & \text{if } a \in K^\times, \\ 0 \mapsto \infty \end{cases}$$

is the **valuation** on K with value group G, valuation ring $R = \{a \in K \mid v(a) \geq 0\}$ and valuation ideal $I = \{a \in K \mid v(a) > 0\}$.

Valuations

Examples

- For any field K the trivial valuation $v\colon K^\times \to \{0\}$ has valuation ring K and residue field K.
- The field of Laurent series $k((x))$ with coefficient field k can be equipped with the valuation
$$v_{\min}\left(\sum_{\ell=m}^{\infty} a_\ell x^\ell\right) = m$$
(where $a_m \neq 0$) with value group $\mathbb{Z}$, valuation ring $k[\![x]\!]$ and residue field k.
- For any totally ordered field $(K,+,-,\cdot,0,1,<)$, any convex subring of K is a valuation ring of K. The finest convex valuation ring on K produces an archimedean residue field.

Henselian Valuations

Definition

Let $(K, +, -, \cdot, 0, 1)$ be a field. A valuation v with valuation ring R on K is **henselian** if for any polynomial

$$p(x) = a_n x^n + \ldots + a_0 \in R[x]$$

and any simple zero $a \in \overline{K}$ of

$$\overline{p}(x) = \overline{a_n} x^n + \ldots + \overline{a_0} \in \overline{K}[x],$$

there exists a zero $b \in R$ of $p(x)$ with $\overline{b} = a$.

Motto: *Simple zeros can be lifted from the residue field.*

Henselian Valuation Rings

Examples

- Any trivial valuation is henselian, as the residue field coincides with the field.
- The valuation $v_{\min}$ on $k((x))$ is henselian.
- If $(K, +, -, \cdot, 0, 1, <)$ is a real closed ordered field, any valuation on K with convex valuation ring is henselian.

1. Ordered Abelian Groups

2. Henselian Valuations

3. Definability

Model Theoretic Setting

- first-order languages $\mathcal{L}$:
  $\mathcal{L}_{\mathrm{r}} = (+, -, \cdot, 0, 1)$, $\mathcal{L}_{\mathrm{or}} = (+, -, \cdot, 0, 1, <)$
- $\mathcal{L}$-structures:
  $(\mathbb{Z}, +, -, \cdot, 0, 1)$, $(\mathbb{R}, +, -, \cdot, 0, 1, <)$
- $\mathcal{L}$-formulas and $\mathcal{L}$-sentences:
  $\forall x \exists y \; x + y = 0$
- $\mathcal{L}$-definability (*without* and *with* parameters):
  $\exists y \; x \cdot y = 1$ defines $\{1, -1\}$ in $(\mathbb{Z}, +, -, \cdot, 0, 1)$ and $\mathbb{R} \setminus \{0\}$ in $(\mathbb{R}, +, -, \cdot, 0, 1, <)$.
  $x \cdot x < \sqrt{3}$ defines $(-\sqrt[4]{3}, \sqrt[4]{3})$ in $(\mathbb{R}, +, -, \cdot, 0, 1, <)$.

Main Questions

A valuation v is said to be $\mathcal{L}$-**definable** if its valuation ring R is $\mathcal{L}$-definable.

Let $(K, +, -, \cdot, 0, 1)$ be a field and let v be a henselian valuation on K.

What are necessary and sufficient conditions such that v is $\mathcal{L}_{\mathrm{r}}$-definable?

Main Questions

A valuation v is said to be $\mathcal{L}$-**definable** if its valuation ring R is $\mathcal{L}$-definable.

Let $(K, +, -, \cdot, 0, 1, <)$ be an ordered field and let v be a henselian valuation on K.

What are necessary and sufficient conditions such that v is $\mathcal{L}_{\mathrm{or}}$-definable?

Note: In a real closed field $(K, +, -, \cdot, 0, 1, <)$, no non-trivial henselian and thus convex valuation is $\mathcal{L}_{\mathrm{or}}$-definable. (Due to o-minimality.)

Motivation: Decidability

Theorem (Ax, 1965)

Let k be a field. Then $v_{\min}$ is (parameter-free) $\mathcal{L}_{\mathrm{r}}$ definable in $k((x))$.

Since the residue field of $v_{\min}$ is k, Ax concludes: If $(k, +, -, \cdot, 0, 1)$ is undecidable, then $(k((x)), +, -, \cdot, 0, 1)$ is undecidable.

Motivation: Classification of NIP Fields

What are necessary and sufficient conditions on a field (an ordered field) to be NIP?

Examples of NIP structures:

- o-minimal structures
- weakly o-minimal structures
- C-minimal structures
- ...

Motivation: Classification of NIP Fields

Shelah–Hasson Conjecture

Any infinite NIP field $(K, +, -, \cdot, 0, 1)$ is either algebraically closed, real closed or admits a non-trivial $\mathcal{L}_{\mathrm{r}}$-definable henselian valuation.

Refinements of the property 'NIP':

dp-minimal $\to$ **dp-finite**
$\to$ **strongly NIP** $\to$ **NIP**

Currently the Shelah–Hasson Conjecture has been verified for the dp-finite case (Johnson, 2020).

Main Questions – Revised

Let $(K, +, -, \cdot, 0, 1)$ be a field and let v be a henselian valuation on K.

What are necessary and sufficient conditions on the residue field and the value group such that v is $\mathcal{L}_{\mathrm{r}}$-definable?

We concentrate on the value group.

Main Questions – Revised

Let $(K, +, -, \cdot, 0, 1, <)$ be an ordered field and let v be a henselian valuation on K.

What are necessary and sufficient conditions on the residue field and the value group such that v is $\mathcal{L}_{\mathrm{or}}$-definable?

We concentrate on the value group.

Discretely Ordered Case

Theorem (Hong, 2013/2014)

Let $(K, +, -, \cdot, 0, 1)$ be a field and let v be a henselian valuation on K. Suppose that its value group G is discretely ordered. Then v is $\mathcal{L}_{\mathrm{r}}$-definable (using one parameter).

- One can generally not obtain definability without parameters, even if one allows the richer language $\mathcal{L}_{\mathrm{or}}$ (K., Kuhlmann, Link, 2021).
- However, if $(G, +, -, 0, <)$ is already elementarily equivalent to $(\mathbb{Z}, +, -, 0, <)$, then v is already $\mathcal{L}_{\mathrm{r}}$-definable *without parameters*.

Densely Ordered Case

Theorem (K., Kuhlmann, Link, 2021)

Let $(K, +, -, \cdot, 0, 1)$ be a field and let v be a henselian valuation on K. Suppose that its value group G is not closed in G^{div}. Then v is $\mathcal{L}_{\mathrm{r}}$-definable (using one parameter).

- Again, one can generally not obtain definability without parameters, even if one allows the richer language $\mathcal{L}_{\mathrm{or}}$. (K., Kuhlmann, Link, 2021)
- However, if G is non-divisible but dense in G^{div}, then one obtains $\mathcal{L}_{\mathrm{r}}$-definability without parameters. (E.g. Hong, 2013/2014, previously shown by Koenigsmann.)

References

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