Generalised power series determined by linear recurrence relations

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Notations

Throughout, we fix the following notations:

- G – (additive) ordered abelian group
- k – field
- $\mathcal{F}$ – family of well-ordered subsets of G

Hahn fields

- The **maximal Hahn field** $k((G))$ consists of all $s\colon G \to k$ with well-ordered support $\mathrm{supp}(s) = \{g \in G \mid s(g) \neq 0\}$. We express $s \in k((G))$ by

$$s = \sum_{g \in G} s_g t^g$$

and thus regard it as a **(generalised) power series**.

Hahn fields

- The **minimal Hahn field** $k(G)$ is the subset of $k((G))$ containing all elements of the form
$$\frac{p(t^{g_1}, \ldots, t^{g_n})}{q(t^{g_1}, \ldots, t^{g_n})}$$
for some $n \in \mathbb{N}$, $p, q \in k[X_1, \ldots, X_n]$, $g_1, \ldots, g_n \in G$ with $q(t^{g_1}, \ldots, t^{g_n}) \neq 0$.

  Note that $k(G)$ is the smallest subfield of $k((G))$ containing all monomials αt^h, where $\alpha \in k$ and $h \in G$.
- A field K with $k(G) \subseteq K \subseteq k((G))$ is called a **Hahn field**.

Recognising $k(G)$ within $k((G))$

Question 1

Given a power series
$$s = \sum_{g\in G} s_g t^g \in k((G)),$$
under what conditions on the support $\mathrm{supp}(s)$ and the coefficients s_g of s is s already contained in $k(G)$?

An answer is known for the case $G = \mathbb{Z}$ (fields of Laurent series).

Canonical lifting property

Given an automorphism $\rho\colon k \to k$ (as field) and an automorphism $\tau\colon G \to G$ (as ordered group), the **canonical lifting** of (ρ, τ) to $k((G))$ is given by

$$\sigma\colon \sum_{g \in G} s_g t^g \mapsto \sum_{g \in G} \rho(s_g) t^{\tau(g)}.$$

A Hahn field K has the **canonical lifting property** if it is closed under the canonical lifting of any pair of automorphisms.

Question 2

Find Hahn fields *with* and Hahn fields *without* the canonical lifting property.

Note that both $k(G)$ and $k((G))$ have the canonical lifting property.

Definition

The field of **Laurent series** with coefficient field k is given by

$$k((t)) := k((\mathbb{Z})).$$

An element $s \in k((t))$ is of the form

$$s = \sum_{i=\ell}^{\infty} s_i t^i$$

for some $\ell \in \mathbb{Z}$.

Rational functions

Fact (Kronecker, 1881)

Let

$$s = \sum_{i=\ell}^{\infty} s_i t^i \in k((t)).$$

Then the following are equivalent:

1. $s \in k(t)$.
2. There exist $m \in \mathbb{N}$ and $c_1, \ldots, c_m \in k$ such that for any $n > m$ the linear recurrence relation with constant coefficients

$$s_n = \sum_{j=1}^{m} c_j s_{n-j}$$

holds.

Rational functions

Fact (revised)

Let

$$s = \sum_{i=\ell}^{\infty} s_i t^i \in k((t)).$$

Then the following are equivalent:

1. $s \in k(t)$.
2. There exist $m \in \mathbb{N}$, $n_0, \ldots, n_m \in \mathbb{Z}$ with $n_0 < \ldots < n_m$ and $c_0, \ldots, c_m \in k$, not all equal to 0, such that for any $n \in \mathbb{Z} \setminus \{n_0, \ldots, n_m\}$ the linear recurrence relation with constant coefficients

$$\sum_{j=0}^{m} c_j s_{n-j} = 0$$

holds.

Example

Let $s = \frac{1+t}{1-t} \in k((t))$. Then

$$s = (1+t)\sum_{i=0}^{\infty} t^i = 1t^0 + \sum_{i=1}^{\infty} 2t^i.$$

Now $m = 1$, $n_0 = 0$, $n_1 = 1$, $c_0 = 1$ and $c_1 = -1$ witness that $s \in k(t)$. Indeed, for any $n \in \mathbb{Z} \setminus \{0, 1\}$ the linear recurrence relation

$$s_n - s_{n-1} = 0$$

holds.

Linear recurrence relations

Hence, within fields of Laurent series, elements of $k(t)$ can be recognised within $k((t))$ by (non-trivial) **linear recurrence relations**.

We establish a notion of linear recurrence relations for generalised power series.

Linear recurrence sequences

Definition

A **(generalised) linear recurrence sequence** in $k((G))$ is a partial function r from G to k whose domain $\mathrm{dom}(r)$ is well-ordered. It is **non-trivial** if there exists $g \in \mathrm{dom}(r)$ with $r(g) \neq 0$.

- Any linear recurrence sequence r associates to a power series

$$r^* := \sum_{g \in \mathrm{dom}(r)} r(g) t^g.$$

- Any power series s associates to a linear recurrence sequence

$$s^* \colon \mathrm{supp}(s) \to k, g \mapsto s(g).$$

Determined sets

Let r be a linear recurrence sequence. Then the order-type α of its domain is an ordinal and we may enumerate r as $r = (g_i, r_i)_{i<\alpha}$, where $\operatorname{dom}(r) = \{g_i \mid i < \alpha\}$ and $r_i = r(g_i)$.

Definition

Let $r = (g_i, r_i)_{i<\alpha}$ be a linear recurrence sequence. We define $\langle r \rangle$ to be the set of all $s \in k((G))$ such that for any $h \in G \setminus \operatorname{dom}(r)$ the following (generalised) linear recurrence relation holds:

$$\sum_{i<\alpha} r(g_i) s_{h-g_i} = 0.$$

For any set R of linear recurrence sequences, we set

$$\langle R \rangle = \bigcup_{r \in R} \langle r \rangle.$$

Example

Part 1: Let $A = \{g_\beta \mid \beta < \alpha\}$ be a well-ordered subset of G of order-type α with $0 \in A$. Set $\mathrm{dom}(r_A) = A$, $r_A(0) = 1$ and $r_A(g) = 0$ for any $g \in A \setminus \{0\}$. Moreover, let $\beta < \alpha$ with $g_\beta = 0$.

Set determined by r_A: We have $s \in \langle r_A \rangle$ if and only if for any $h \in G \setminus A$:

$$0 = \sum_{i<\alpha} r_A(g_i) s_{h-g_i} = r_A(g_\beta) s_{h-g_\beta} = s_h.$$

We obtain

$$\langle r_A \rangle = \{s \in k((G)) \mid \mathrm{supp}(s) \subseteq A\}.$$

Part 2: Setting R_{fin} to be the set of all r_A where A is a finite subset of G containing 0, we obtain

$$\langle R_{\mathrm{fin}} \rangle = k[G] := \{p(t^{g_1}, \dots, t^{g_n}) \mid n \in \mathbb{N}, p \in k[X_1, \dots, X_n], g_1, \dots, g_n \in G\}.$$

Main Lemma

Proposition

For any trivial linear recurrence sequence r, we have $\langle r\rangle = k((G))$.

Main Lemma

Let r be a non-trivial linear recurrence sequence. Then

$$\langle r\rangle = \left\{ \frac{s}{r^*} \;\middle|\; s \in k((G)), \operatorname{supp}(s) \subseteq \operatorname{dom}(r) \right\}.$$

Hence, $\langle r\rangle$ is a k-vector space containing k.

However, $\langle R\rangle$ is not in general closed under addition.

Example

Consider

$$s = 1 + 1t + 2t^2 + 3t^3 + 5t^4 + 8t^5 + 13t^6 + \ldots \in k((\mathbb{Z})).$$

For any $n \in \mathbb{Z} \setminus \{0, 1, 2\}$, the linear recurrence relation

$$s_n - s_{n-1} - s_{n-2} = 0$$

holds. By our Main Lemma, $s \in \left\{ \frac{a+bt+ct^2}{1-t-t^2} \;\middle|\; a, b, c \in k \right\}$. Indeed,

$$s = \frac{1}{1 - t - t^2} \in k(\mathbb{Z}).$$

Example

Consider

$$s = 1 + 1t^{\sqrt{2}} + 2t^{2\sqrt{2}} + 3t^{3\sqrt{2}} + 5t^{4\sqrt{2}} + 8t^{5\sqrt{2}} + 13t^{6\sqrt{2}} + \ldots \in k((\mathbb{R})).$$

For any $h \in \mathbb{R} \setminus \{0, \sqrt{2}, 2\sqrt{2}\}$, the linear recurrence relation

$$s_h - s_{h-\sqrt{2}} - s_{h-2\sqrt{2}} = 0$$

holds. By our Main Lemma, $s \in \left\{ \frac{a+bt^{\sqrt{2}}+ct^{2\sqrt{2}}}{1-t^{\sqrt{2}}-t^{2\sqrt{2}}} \;\middle|\; a, b, c \in k \right\}$. Indeed,

$$s = \frac{1}{1 - t^{\sqrt{2}} - t^{2\sqrt{2}}} \in k(\mathbb{R}).$$

Determined fields

Proposition

Let R be a non-empty set of non-trivial linear recurrence sequences satisfying the following:

1. For any $r \in R$, any other non-trivial linear recurrence with domain $\mathrm{dom}(r)$ also lies in R.
2. For any $r_1, r_2 \in R$, any non-trivial linear recurrence with domain $\mathrm{dom}(r_1) + \mathrm{dom}(r_2) = \{h_1 + h_2 \mid h_1 \in \mathrm{dom}(r_1), h_2 \in \mathrm{dom}(r_2)\}$ also lies in R.

Then $\langle R \rangle$ is a subfield of $k((G))$ containing k.

Idea: Let R consist of all non-trivial linear recurrence sequences whose domain lies in a given family $\mathcal{F}$ of well-ordered subsets of G.

Definition

Recall that $\mathcal{F}$ denotes a family of well-ordered subsets of G.

Definition

An $\mathcal{F}$-**sequence** is a non-trivial linear recurrence sequence whose domain lies in $\mathcal{F}$. We denote by $S(\mathcal{F})$ the set of all $\mathcal{F}$-sequences.

Corollary

Suppose that $\mathcal{F}$ is non-empty and closed under sums, i.e. for any $A, B \in \mathcal{F}$ also $A + B \in \mathcal{F}$. Then $\langle S(\mathcal{F}) \rangle$ is a subfield of $k((G))$ containing k.

Determined Hahn fields

Proposition

Let $\mathcal{F}_{\mathrm{fin}}$ be the family of all finite subsets of G. Then

$$\langle S(\mathcal{F}_{\mathrm{fin}})\rangle = k(G).$$

Thus, elements of $k(G)$ (the field of fractions of $k[G]$) are determined by linear recurrence relations.

Corollary

Suppose that $\mathcal{F}$ is closed under sums and contains $\mathcal{F}_{\mathrm{fin}}$. Then $\langle S(\mathcal{F})\rangle$ is a Hahn field.

Example: non-determined Hahn field

Not every Hahn field is determined by linear recurrence relations!

For instance, let K be the relative algebraic closure of $\mathbb{Q}(\mathbb{Z})$ inside $\mathbb{Q}((\mathbb{Z}))$. (Or take any other countable ordered abelian group instead of $\mathbb{Z}$.) Then K is countable.

However, any field determined by linear recurrence relations strictly containing $\mathbb{Q}(\mathbb{Z})$ must be uncountable.

Indeed, if R contains a linear recurrence sequence r with infinite domain, then

$$R \supseteq \langle r \rangle = \left\{ \frac{s}{r^*} \;\middle|\; s \in k((G)), \operatorname{supp}(s) \subseteq \operatorname{dom}(r) \right\}.$$

k-hulls

Definition

We define the k-**hull** of $\mathcal{F}$ as the set

$$k((\mathcal{F})) = \{s \in k((G)) \mid \operatorname{supp}(s) \in \mathcal{F}\}.$$

Definition

Suppose that $\mathcal{F}$ satisfies the following:

- $\mathcal{F} \neq \emptyset$ and $\bigcup_{A \in \mathcal{F}} A$ generates G as a group.
- If $B \subseteq A \in \mathcal{F}$, then $B \in \mathcal{F}$.
- If $A, B \in \mathcal{F}$, then $A \cup B \in \mathcal{F}$.
- If $A \in \mathcal{F}$ and $g \in G$, then $A + \{g\} \in \mathcal{F}$.
- If $A \in \mathcal{F}$ with $A \subseteq G^{\geq 0}$, then $\{\sum_{i=1}^{n} a_i \mid n \in \mathbb{N} \cup \{0\}, a_1, \ldots, a_n \in A\} \in \mathcal{F}$.

Then $\mathcal{F}$ is called a **Rayner field family** and $k((\mathcal{F}))$ is called its **Rayner field**.

Hahn and Rayner fields

Theorem (K., Kuhlmann, Serra; 2022)

Suppose that $\mathrm{char}(k) = 0$. Then $k((\mathcal{F}))$ is a Hahn field if and only if it is a Rayner fields.

Are all Rayner fields determined by linear recurrence relations?

Determined Rayner fields

For a well-ordered subset A of G, we set

$$r_A \colon A \cup \{0\} \to k, g \mapsto \begin{cases} 1, & \text{if } g = 0, \\ 0, & \text{if } g \in A \setminus \{0\}. \end{cases}$$

Moreover, we set

$$R_{\mathcal{F}} = \{r_A \mid A \in \mathcal{F}\}.$$

Proposition

Suppose that $\mathcal{F}$ is closed under subsets and unions with $\{0\}$. Then $\langle R_{\mathcal{F}} \rangle = k((\mathcal{F}))$. In particular, if $\mathcal{F}$ is a Rayner field family, then $\langle R_{\mathcal{F}} \rangle$ is its Rayner field.

Task

Find Hahn fields *with* and Hahn fields *without* the canonical lifting property (CLP).

Rayner fields without the CLP

Proposition (Kuhlmann, Serra; 2022)

A Rayner field $k((\mathcal{F}))$ has the canonical lifting property if and only if $\mathcal{F}$ is stable under automorphisms on G, i.e. for any automorphism τ on G and any $A \in \mathcal{F}$ also $\tau(A) \in \mathcal{F}$.

Constructing a Rayner field family $\mathcal{F}$ that is not stable under automorphisms on G:

Let $G = \coprod_{\mathbb{Z}} \mathbb{Q}$ and let $A = \{-1/p_i \cdot \mathbb{1}_1 \mid i \in \mathbb{N}\}$, where p_i denotes the i-th prime number. Let $\mathcal{F}$ consist of all well-ordered subsets of subgroups of G of the form

$$\langle g_1, \ldots, g_n, A \rangle,$$

where $g_1, \ldots, g_n \in G$. Then $\mathcal{F}$ is not stable under the automorphism on G induced by

$$\mathbb{1}_n \mapsto \mathbb{1}_{n-1}$$

for any $n \in \mathbb{Z}$.

Hahn fields with the CLP determined by $\mathcal{F}$-sequences

Theorem

Suppose that $\mathcal{F}$ satisfies the following:

- $\mathcal{F}_{\text{fin}} \subseteq \mathcal{F}$.
- $\mathcal{F}$ is closed under sums.
- $\mathcal{F}$ is closed under automorphisms on G.

Then $\langle S(\mathcal{F}) \rangle$ is a Hahn field with the canonical lifting property.

This gives a construction methods for Hahn field with the canonical lifting property via families of well-ordered subsets of $\mathcal{F}$.

Thank you for your attention!

References

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