O-minimal Exponential Fields and Peano Arithmetic

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Model Theoretic Setting

- first-order languages $\mathcal{L}_{\mathrm{or}} = \{+, -, \cdot, 0, 1, <\}$ and $\mathcal{L}_{\mathrm{exp}} = \{+, -, \cdot, 0, 1, <, \exp\}$
- structures in the languages

  e.g. ordered fields $(K, +, -, \cdot, 0, 1, <)$ or the real exponential field $\mathbb{R}_{\mathrm{exp}} = (\mathbb{R}, +, -, \cdot, 0, 1, <, \exp)$
- formulas and sentences

  e.g. the $\mathcal{L}_{\mathrm{or}}$-formula $\exists x\ x^2 + yx + 1 = 0$ or the $\mathcal{L}_{\mathrm{exp}}$-sentence $\forall x \exists y\ \exp(\exp(x)) < \exp(x + y)$
- (complete) first-order theories

  e.g. the $\mathcal{L}_{\mathrm{or}}$-theory of real closed fields $\mathrm{Th}(\mathbb{R}, +, -, \cdot, 0, 1, <)$ consisting of all $\mathcal{L}_{\mathrm{or}}$-sentences true over $\mathbb{R}$
- definable sets

  e.g. the $\mathcal{L}_{\mathrm{exp}}$-formula $\exists y\ \exp(y) = x + \pi$ defines the set $(-\pi, \infty)$ in $\mathbb{R}_{\mathrm{exp}}$

Tarski's Quantifier Elimination

Theorem (Tarski, 1948)

For any $\mathcal{L}_{\mathrm{or}}$-formula φ there exists a quantifier-free $\mathcal{L}_{\mathrm{or}}$-formula ψ such that φ and ψ are equivalent over $\mathbb{R}$.

Example: The $\mathcal{L}_{\mathrm{or}}$-formula $\exists x\ x^2 + yx + 1 = 0$ is equivalent over $\mathbb{R}$ to $y^2 - 4 \geq 0$.

Tarski proved this theorem by presenting an explicit quantifier elimination algorithm.

Consequences of Tarski's Quantifier Elimination

- The $\mathcal{L}_{\mathrm{or}}$-theory of $\mathbb{R}$ is decidable, i.e. there exists an algorithm which decides whether a given $\mathcal{L}_{\mathrm{or}}$-sentence is true or false in $\mathbb{R}$.
- Every $\mathcal{L}_{\mathrm{or}}$-definable subset of $\mathbb{R}^n$ is a semi-algebraic set.
- In particular, any $\mathcal{L}_{\mathrm{or}}$-definable subset of $\mathbb{R}$ is a finite union of points and open intervals.
- **Tarski's Transfer Principle:** Any $\mathcal{L}_{\mathrm{or}}$-sentence which is true in $\mathbb{R}$ is also true in any real closed field.

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Tarski's Exponential Function Problem: Is $\mathbb{R}_{\mathrm{exp}}$ also decidable?

Schanuel's Conjecture

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Let $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ be linearly independent over $\mathbb{Q}$. Then

$$\mathrm{td}_{\mathbb{Q}}(\mathbb{Q}(\alpha_1, \ldots, \alpha_n, \mathrm{e}^{\alpha_1}, \ldots, \mathrm{e}^{\alpha_n})) \geq n.$$

→ Schanuel's Conjecture would, for instance, imply the algebraic independence of e and π.

Real Schanuel's Conjecture (SC)

Let $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ be linearly independent over $\mathbb{Q}$. Then

$$\mathrm{td}_{\mathbb{Q}}(\mathbb{Q}(\alpha_1, \ldots, \alpha_n, \mathrm{e}^{\alpha_1}, \ldots, \mathrm{e}^{\alpha_n})) \geq n.$$

Decidability of the Real Exponential Field

Decidability Conjecture

The real exponential field $\mathbb{R}_{\text{exp}}$ is decidable.

Theorem (Macintyre and Wilkie, 1996)

Assume (SC). Then $\mathbb{R}_{\text{exp}}$ is decidable.

Schanuel's Conjecture

$\Longrightarrow$

Decidability Conjecture

O-minimal Exponential Fields

Definition

Let K be an ordered field. A unary function exp which is an order-preserving isomorphism from $(K, +, 0, <)$ to $(K^{>0}, \cdot, 1, <)$ is called an **exponential** on K. The $\mathcal{L}_{\exp}$-structure $(K, \exp)$ is called an **ordered exponential field**.

Definition

A linearly ordered structure $(M, <, \ldots)$ is called **o-minimal** if every definable subset of M is a finite union of points and open intervals in M.

Theorem (Wilkie, 1996)

The real exponential field $\mathbb{R}_{\exp}$ is o-minimal.

Example: The $\mathcal{L}_{\exp}$-formula $\exists y\ x^2 > \exp(y) + \pi$ defines the set $(-\infty, -\sqrt{\pi}) \cup (\sqrt{\pi}, \infty)$ over $\mathbb{R}_{\exp}$.

Transfer Conjecture

- **Elementary equivalence:** Two $\mathcal{L}$-structures $\mathcal{M}$ and $\mathcal{N}$ are elementarily equivalent if they satisfy exactly the same $\mathcal{L}$-sentences. We write $\mathcal{M} \equiv \mathcal{N}$.
- **EXP:** $\mathcal{L}_{\mathrm{exp}}$-sentence stating that the differential equation $\exp' = \exp$ with initial condition $\exp(0) = 1$ holds.

Transfer Conjecture (TC)

Let $(K, \exp)$ be an o-minimal EXP-field. Then $(K, \exp) \equiv \mathbb{R}_{\mathrm{exp}}$.

Theorem (Berarducci and Servi, 2004)

Assume (TC). Then $\mathbb{R}_{\mathrm{exp}}$ is decidable.

Schanuel's Conjecture ⇒ Decidability Conjecture ⇐ Transfer Conjecture

Resulting Questions

Schanuel's Conjecture ⇒ Decidability Conjecture ⇐ Transfer Conjecture

- What are the connections between Schanuel's Conjecture and the Transfer Conjecture?
- What properties of $\mathbb{R}_{\text{exp}}$ can be generalised to any o-minimal EXP-field?
- What are construction methods for o-minimal EXP-fields?

Connections between Schanuel's Conjecture and Transfer Conjecture

- Assuming (SC), any o-minimal EXP-field satisfies the existential theory $\mathsf{Th}_\exists(\mathbb{R}_{\mathsf{exp}})$ of $\mathbb{R}_{\mathsf{exp}}$.
- Assuming (TC), if some o-minimal EXP-field satisfies Schanuel's Conjecture, then all o-minimal EXP-fields do so.

Properties of $\mathbb{R}_{\exp}$ Generalised to O-minimal EXP-fields

- Several analytic properties of the exponential function, such as Taylor approximation or exponential growth, hold in any o-minimal EXP-field.
- For any o-minimal EXP-field $(K, \exp)$, we have $(\overline{K}, \overline{\exp}) \preceq \mathbb{R}_{\exp}$. Here, $\overline{K}$ is the residue field of K under the natural valuation and $\overline{\exp}$ is the exponential induced on the residue field.

Constructions of O-minimal EXP-fields

- Starting with certain **countable archimedean fields** F and **countable divisible ordered abelian groups** G (both with additional structure), we construct countable models of real exponentiation $(K, \exp)$ with **residue field** F and **value group** G under the natural valuation.
- Starting with an **arbitrary o-minimal** EXP**-field** $(K, \exp)$, we construct an exponential exp on the real closed field of **surreal numbers No** with $(K, \exp) \preceq (\mathbf{No}, \exp)$.
- Starting with certain models M of **Peano Arithmetic**, we construct o-minimal EXP-fields with **integer part** $M \cup (-M)$.

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Fragments of Arithmetic

- first-order language $\mathcal{L}_{\mathrm{PA}} = \{+, \cdot, 0, 1, <\}$
- PA^-: axiomatisation of discretely ordered commutative semirings with modified subtraction, 0 as least element and 1 as its direct successor
- **Peano Arithmetic** PA: extension of PA^- by induction scheme for every $\mathcal{L}_{\mathrm{PA}}$-formula
- **Open Induction** IOpen: extension of PA^- by induction scheme for every *quantifier-free* $\mathcal{L}_{\mathrm{PA}}$-formulas
- **True Arithmetic**: complete first-ordered $\mathcal{L}_{\mathrm{PA}}$-theory of $\mathbb{N}$

$$\mathrm{PA}^- \subsetneq \mathrm{IOpen} \subseteq \textit{further fragments of arithmetic} \subseteq \mathrm{PA} \subsetneq \mathrm{Th}(\mathbb{N}, +, \cdot, 0, 1, <)$$

Integer Parts

Integer part: a discretely ordered subring Z of ordered field K such that for any $a \in K$ there exists a unique $k \in Z$ with $k \leq a < k+1$

e.g. $\mathbb{Z}$ is an integer part of $\mathbb{Q}$ and of $\mathbb{R}$

Theorem (Shepherdson, 1964)

Let M be a model of PA^-. Then the following are equivalent:

1. *$M \cup (-M)$ is an integer part of a real closed field.*
2. *M is a model of IOpen.*

General Questions

1. Let K be an ordered field and let Z be an integer part of K. What algebraic and model theoretic information of K is already "encoded" in Z?
2. Let Z be a discretely ordered ring such that $Z^{\geq 0}$ models a fragment of arithmetic. Is there an ordered field K having Z as an integer part and exhibiting good algebraic and model theoretic properties?

General Questions

1. Let K be an ordered field and let Z be an integer part of K. What algebraic and model theoretic information of K is already "encoded" in Z?
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M-reals

Let M be a model of PA.

- M-**integers**: $\mathbb{Z}_M = M \cup (-M)$.
- M-**rationals**: $\mathbb{Q}_M$ is the field of fractions of $\mathbb{Z}_M$.
- M-**reals**: K_M consists of equivalence classes of rational M-Cauchy sequences $a\colon M \to \mathbb{Q}_M, n \mapsto a_n$ definable over $\mathbb{Q}_M$.

By this construction, $\mathbb{Z}_M$ is an integer part of K_M with $\mathbb{Z}_M^{\geq 0} = M$ and $\mathbb{Q}_M$ is dense in K_M.

Main Results (Carl, K.; 2021)

Theorem (Models of PA)

Let M be a model of Peano Arithmetic. Then K_M can be endowed with an exponential E such that (K_M, E) is a real closed EXP-field. Moreover, if (K_M, E) is model complete, then it is o-minimal.

Theorem (Models of $\mathrm{Th}(\mathbb{N}, +, \cdot, 0, 1, <)$)

Let M be a model of True Arithmetic. Then $(K_M, E) \equiv \mathbb{R}_{\exp}$.

Theorem (IPA-RCFs)

Let K be a countable non-archimedean real closed field with an integer part Z such that $Z^{\geq 0}$ is a model of Peano Arithmetic. Then K admits an exponential.

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