

O-minimal Exponential Fields and Peano Arithmetic

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27 September 2022

Colloquium Logicum 2022, PhD Colloquium

- 1 Motivating Conjectures
- 2 Overview of Main Results
- 3 Models of Peano Arithmetic

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Model Theoretic Setting

- first-order languages $\mathcal{L}_{\text{or}} = \{+, -, \cdot, 0, 1, <\}$ and $\mathcal{L}_{\text{exp}} = \{+, -, \cdot, 0, 1, <, \exp\}$
- structures in the languages
e.g. ordered fields $(K, +, -, \cdot, 0, 1, <)$ or the real exponential field $\mathbb{R}_{\text{exp}} = (\mathbb{R}, +, -, \cdot, 0, 1, <, \exp)$
- formulas and sentences
e.g. the \mathcal{L}_{or} -formula $\exists x \ x^2 + yx + 1 = 0$ or the \mathcal{L}_{exp} -sentence $\forall x \exists y \ \exp(\exp(x)) < \exp(x + y)$
- (complete) first-order theories
e.g. the \mathcal{L}_{or} -theory of real closed fields $\text{Th}(\mathbb{R}, +, -, \cdot, 0, 1, <)$ consisting of all \mathcal{L}_{or} -sentences true over \mathbb{R}
- definable sets
e.g. the \mathcal{L}_{exp} -formula $\exists y \ \exp(y) = x + \pi$ defines the set $(-\pi, \infty)$ in \mathbb{R}_{exp}

Tarski's Quantifier Elimination

Theorem (Tarski, 1948)

For any \mathcal{L}_{or} -formula φ there exists a quantifier-free \mathcal{L}_{or} -formula ψ such that φ and ψ are equivalent over \mathbb{R} .

Example: The \mathcal{L}_{or} -formula $\exists x \ x^2 + yx + 1 = 0$ is equivalent over \mathbb{R} to $y^2 - 4 \geq 0$.

Tarski proved this theorem by presenting an explicit quantifier elimination algorithm.

Consequences of Tarski's Quantifier Elimination

- The \mathcal{L}_{or} -theory of \mathbb{R} is decidable, i.e. there exists an algorithm which decides whether a given \mathcal{L}_{or} -sentence is true or false in \mathbb{R} .
- Every \mathcal{L}_{or} -definable subset of \mathbb{R}^n is a semi-algebraic set.
- In particular, any \mathcal{L}_{or} -definable subset of \mathbb{R} is a finite union of points and open intervals.
- **Tarski's Transfer Principle:** Any \mathcal{L}_{or} -sentence which is true in \mathbb{R} is also true in any real closed field.

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Tarski's Exponential Function Problem: Is \mathbb{R}_{exp} also decidable?

Schanuel's Conjecture

Schanuel's Conjecture

Let $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ be linearly independent over \mathbb{Q} . Then

$$\text{td}_{\mathbb{Q}}(\mathbb{Q}(\alpha_1, \dots, \alpha_n, e^{\alpha_1}, \dots, e^{\alpha_n})) \geq n.$$

→ Schanuel's Conjecture would, for instance, imply the algebraic independence of e and π .

Real Schanuel's Conjecture (SC)

Let $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ be linearly independent over \mathbb{Q} . Then

$$\text{td}_{\mathbb{Q}}(\mathbb{Q}(\alpha_1, \dots, \alpha_n, e^{\alpha_1}, \dots, e^{\alpha_n})) \geq n.$$

Decidability of the Real Exponential Field

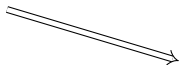
Decidability Conjecture

The real exponential field \mathbb{R}_{exp} is decidable.

Theorem (Macintyre and Wilkie, 1996)

Assume (SC). Then \mathbb{R}_{exp} is decidable.

Schanuel's Conjecture



Decidability Conjecture

O-minimal Exponential Fields

Definition

Let K be an ordered field. A unary function \exp which is an order-preserving isomorphism from $(K, +, 0, <)$ to $(K^{>0}, \cdot, 1, <)$ is called an **exponential** on K . The \mathcal{L}_{\exp} -structure (K, \exp) is called an **ordered exponential field**.

Definition

A linearly ordered structure $(M, <, \dots)$ is called **o-minimal** if every definable subset of M is a finite union of points and open intervals in M .

Theorem (Wilkie, 1996)

The real exponential field \mathbb{R}_{\exp} is o-minimal.

Example: The \mathcal{L}_{\exp} -formula $\exists y \ x^2 > \exp(y) + \pi$ defines the set $(-\infty, -\sqrt{\pi}) \cup (\sqrt{\pi}, \infty)$ over \mathbb{R}_{\exp} .

Transfer Conjecture

- **Elementary equivalence:** Two \mathcal{L} -structures \mathcal{M} and \mathcal{N} are elementarily equivalent if they satisfy exactly the same \mathcal{L} -sentences. We write $\mathcal{M} \equiv \mathcal{N}$.
- **EXP:** \mathcal{L}_{exp} -sentence stating that the differential equation $\text{exp}' = \text{exp}$ with initial condition $\text{exp}(0) = 1$ holds.

Transfer Conjecture (TC)

Let (K, exp) be an o-minimal EXP-field. Then $(K, \text{exp}) \equiv \mathbb{R}_{\text{exp}}$.

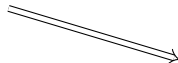
Theorem (Berarducci and Servi, 2004)

Assume (TC). Then \mathbb{R}_{exp} is decidable.

Schanuel's Conjecture

Transfer Conjecture

Decidability Conjecture



Resulting Questions

Schanuel's Conjecture

Transfer Conjecture

Decidability Conjecture

- What are the connections between Schanuel's Conjecture and the Transfer Conjecture?
- What properties of \mathbb{R}_{exp} can be generalised to any o-minimal EXP-field?
- What are construction methods for o-minimal EXP-fields?

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Connections between Schanuel's Conjecture and Transfer Conjecture

- Assuming (SC), any o-minimal EXP-field satisfies the existential theory $\text{Th}_{\exists}(\mathbb{R}_{\text{exp}})$ of \mathbb{R}_{exp} .
- Assuming (TC), if some o-minimal EXP-field satisfies Schanuel's Conjecture, then all o-minimal EXP-fields do so.

Properties of \mathbb{R}_{\exp} Generalised to O-minimal EXP-fields

- Several analytic properties of the exponential function, such as Taylor approximation or exponential growth, hold in any o-minimal EXP-field.
- For any o-minimal EXP-field (K, \exp) , we have $(\overline{K}, \overline{\exp}) \preceq \mathbb{R}_{\exp}$. Here, \overline{K} is the residue field of K under the natural valuation and $\overline{\exp}$ is the exponential induced on the residue field.

Constructions of O-minimal EXP-fields

- Starting with certain **countable archimedean fields** F and **countable divisible ordered abelian groups** G (both with additional structure), we construct countable models of real exponentiation (K, \exp) with **residue field** F and **value group** G under the natural valuation.
- Starting with an **arbitrary o-minimal EXP-field** (K, \exp) , we construct an exponential \exp on the real closed field of **surreal numbers** \mathbf{No} with $(K, \exp) \preceq (\mathbf{No}, \exp)$.
- Starting with certain models M of **Peano Arithmetic**, we construct o-minimal EXP-fields with **integer part** $M \cup (-M)$.

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Fragments of Arithmetic

- first-order language $\mathcal{L}_{\text{PA}} = \{+, \cdot, 0, 1, <\}$
- PA^- : axiomatisation of discretely ordered commutative semirings with modified subtraction, 0 as least element and 1 as its direct successor
- **Peano Arithmetic** PA: extension of PA^- by induction scheme for every \mathcal{L}_{PA} -formula
- **Open Induction** IOpen : extension of PA^- by induction scheme for every *quantifier-free* \mathcal{L}_{PA} -formulas
- **True Arithmetic**: complete first-ordered \mathcal{L}_{PA} -theory of \mathbb{N}

$\text{PA}^- \subsetneq \text{IOpen} \subseteq \text{further fragments of arithmetic} \subseteq \text{PA} \subsetneq \text{Th}(\mathbb{N}, +, \cdot, 0, 1, <)$

Integer Parts

Integer part: a discretely ordered subring Z of ordered field K such that for any $a \in K$ there exists a unique $k \in Z$ with $k \leq a < k + 1$

e.g. \mathbb{Z} is an integer part of \mathbb{Q} and of \mathbb{R}

Theorem (Shepherdson, 1964)

Let M be a model of PA^- . Then the following are equivalent:

- ① *$M \cup (-M)$ is an integer part of a real closed field.*
- ② *M is a model of IOpen .*

General Questions

- 1 Let K be an ordered field and let Z be an integer part of K . What algebraic and model theoretic information of K is already “encoded” in Z ?
- 2 Let Z be a discretely ordered ring such that $Z^{\geq 0}$ models a fragment of arithmetic. Is there an ordered field K having Z as an integer part and exhibiting good algebraic and model theoretic properties?

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M -reals

Let M be a model of PA.

- **M -integers:** $\mathbb{Z}_M = M \cup (-M)$.
- **M -rationals:** \mathbb{Q}_M is the field of fractions of \mathbb{Z}_M .
- **M -reals:** K_M consists of equivalence classes of rational M -Cauchy sequences $a: M \rightarrow \mathbb{Q}_M, n \mapsto a_n$ definable over \mathbb{Q}_M .

By this construction, \mathbb{Z}_M is an integer part of K_M with $\mathbb{Z}_M^{\geq 0} = M$ and \mathbb{Q}_M is dense in K_M .

Main Results (Carl, K.; 2021)

Theorem (Models of PA)

Let M be a model of Peano Arithmetic. Then K_M can be endowed with an exponential E such that (K_M, E) is a real closed EXP-field. Moreover, if (K_M, E) is model complete, then it is o-minimal.

Theorem (Models of $\text{Th}(\mathbb{N}, +, \cdot, 0, 1, <)$)

Let M be a model of True Arithmetic. Then $(K_M, E) \equiv \mathbb{R}_{\text{exp}}$.

Theorem (IPA-RCFs)

Let K be a countable non-archimedean real closed field with an integer part Z such that $Z^{\geq 0}$ is a model of Peano Arithmetic. Then K admits an exponential.

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