BE Mathematical Extended Essay

Constructions of the real numbers

a set theoretical approach

by Lothar Sebastian Krapp

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Date of submission: 24 March 2014

Oxford, United Kingdom, 2014

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Notation and terminology

Throughout the essay, the same symbols will be used to describe different but similar constants, operations, relations or sets. In particular this includes the numbers 0 and 1, the binary operators $(+)$ and $(\cdot)$, the order relation $(<)$, the equivalence relation $(\sim)$ and the sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$. Usually it should be clear from the context which specific constant, operation, relation or set the symbol used refers to. If, however, confusion is likely to arise, subscripts will be used to avoid ambiguity.

The multiplication symbol $(\cdot)$ will sometimes be omitted, and the usual conventions for bracketing and the order of different operations in an equation will be used.

We are only going to work with total orders denoted by $(<)$. We will therefore sometimes refer to totally ordered sets only as *ordered sets*, and to total orders simply as *orders*. The standard notation for the corresponding *weak orders* $(\leq)$ and *converses* $(>)$ and $(\geq)$ will be used.

We are only going to work with commutative rings and semirings with a multiplicative identity and will therefore simply refer to them as *rings* and *semirings* respectively.

1 Introduction

"Please forget everything you have learnt in school; for you have not learnt it."

This advice is given by Landau in *Grundlagen der Analysis* (Foundations of Analysis) [9] to students reading his book. Just like Landau, we will establish the real numbers in a constructive way by defining what a real number is and then deducing further properties, opposed to the synthetic approach which only describes the properties of the set of real numbers and postulates the existence of an algebraic system satisfying these properties. Hence, it is recommendable to forget everything we believe to know about the real numbers so that we can start from scratch.

The set of real numbers $\mathbb{R}$ is widely introduced as the (unique) set satisfying the axioms for a complete ordered field. While different axiomatic descriptions of ordered fields are usually quite similar, there exist various versions of the completeness axiom whose equivalence is non-trivial. A version frequently used is the *supremum property*. It states that every non-empty subset of the real numbers which is bounded above has a least upper bound in the real numbers.

Besides this axiomatic approach, $\mathbb{R}$ can also be constructed from basic principles of set theory. In particular when it comes to completeness, there are several different methods of filling the "gaps" in $\mathbb{Q}$ leading to those various equivalent versions of the completeness axiom.

This essay will firstly demonstrate how $\mathbb{Q}$ can be founded from set theoretical first principles and secondly describe and contrast different ways of constructing the real numbers from the rational numbers.

We will start from the premise that the set of natural numbers and its properties have been established set theoretically.

In the first chapter, a standard way of constructing the integers from the natural numbers and thereupon the rational numbers from the integers will be demonstrated. Both the integers and the rational numbers form algebraic constructs which can be introduced axiomatically. Therefore, in each section the axiomatic approach will briefly be explained, then the set including operations and order relations will be defined, and finally we will give a proof that the set does indeed satisfy the axioms given before.

In the second chapter, two classical approaches to completeness, namely Dedekind's and Cantor's constructions, will be explained in detail. The versions of the completeness axiom corresponding to each construction will be stated.

A non-standard approach via hyperrational numbers is described in the third chapter. Since it is a rather uncommon way of completing the rational numbers, a version of the completeness axiom is suggested but not considered thereafter.

The last chapter will contain a theorem establishing uniqueness of the real numbers and finally discuss how different concepts of completeness are useful, but how they all result in the same object: the complete ordered field of real numbers.

Throughout the essay, once we have shown that a set we have established forms a certain algebraic construct, we will use the standard notations and results for it, e. g. uniqueness of identities and inverses etc.

2 Construction of the rational numbers

Both Goldrei [7] and Landau [9] describe in detail how to construct the real numbers starting from the natural numbers. However, they use different approaches: While Goldrei constructs the integers from the naturals, thereupon the rationals, and ultimately the reals, Landau first constructs the positive rationals from the naturals, then the the positive reals and finally gives a notion of negative real numbers.[1] Since the methods used in [7] consist entirely of set theoretical constructions, we will mainly follow this one. Enderton's construction in [4] is similar. All three of these references will be used for justification of certain results.

2.1 Tools for the natural numbers

The *set of natural numbers* $\mathbb{N}$, as introduced in Pila [12], is the unique inductive set contained in every inductive set. Furthermore, it is equipped with the two binary operators of *addition* $(+)$ and *multiplication* $(\cdot)$, which are recursively defined on $\mathbb{N}$. A *strict total order* on $\mathbb{N}$ is given by

$$n < m :\Longleftrightarrow n \in m.$$

Multiplication and addition are *associative* and *commutative*, multiplication is *distributive* over addition, and 0 and 1 are the *additive* and *multiplicative identity* respectively (see [4] p. 79 ff.). That is, for all $m, n, k \in \mathbb{N}$, the following hold:

$$(m + n) + k = m + (n + k), \tag{2.1}$$

$$m + n = n + m, \tag{2.2}$$

$$n + 0 = n, \tag{2.3}$$

$$(m \cdot n) \cdot k = m \cdot (n \cdot k), \tag{2.4}$$

$$m \cdot n = n \cdot m, \tag{2.5}$$

$$n \cdot 1 = n, \tag{2.6}$$

$$m \cdot (n + k) = (m \cdot n) + (m \cdot k). \tag{2.7}$$

The order relation $(<)$ has the following properties (see [4] p. 84 f.):
For all $m, n, k \in \mathbb{N}$:

$$\text{If } m < n, \text{ then } m + k < n + k. \tag{2.8}$$

$$\text{If } m < n \text{ and } 0 < k, \text{ then } k \cdot m < k \cdot n. \tag{2.9}$$

$$\text{If } m < n \text{ and } n < k, \text{ then } m < k. \tag{2.10}$$

[1] We will see in section 3.1 why avoiding negative real numbers at first makes it easier to define multiplication on the real numbers.

$$\text{Exactly one of the following holds: } m < n,\ m = n, \text{ or } n < m. \tag{2.11}$$

In fact, these properties of $\mathbb{N}$ directly imply that $(\mathbb{N}, +, \cdot, <)$ is an ordered semiring (see Definition A.1). These tools for the natural numbers are enough to define the integers with corresponding operations and order relation.

2.2 From the naturals to the integers

Axiomatically, the set of integers $\mathbb{Z}$ can be introduced as the smallest ordered ring. This means that every ordered ring R contains a natural copy of $\mathbb{Z}$, namely the additive subgroup of R generated by 1_R. The set $\mathbb{N}$ can be considered as the subset of non-negative integers.

However, in this way the negative integers are obtained synthetically by defining them as additive inverses of corresponding natural numbers. We do not obtain a set theoretical definition!

The property of the ring $\mathbb{Z}$ which distinguishes it from the semiring $\mathbb{N}$ is the existence of additive inverses. To define negative integers rigorously, we therefore need, in some sense, a notion of subtraction.

The trick is to consider ordered pairs $(m, n) \in \mathbb{N} \times \mathbb{N}$. Using our traditional notion of subtraction on the integers, the ordered pair (m, n) corresponds to the number $m - n$ in $\mathbb{Z}$. The representation of an integer by a pair of naturals is not unique, as different pairs of natural numbers can correspond to the same integer. For instance, $(0, 1)$ and $(2, 3)$ correspond to $0 - 1$ and $2 - 3$ respectively, which both equal -1. Hence, we must define an equivalence relation by determining when two pairs of natural numbers correspond to the same integer. The idea behind this equivalence relation will be that $m - n = k - \ell$ if and only if $m + \ell = k + n$.

Definition 2.1. We define a relation $(\sim)$ on $\mathbb{N} \times \mathbb{N}$ by

$$(m, n) \sim (k, \ell) :\Longleftrightarrow m + \ell = k + n.$$

Notation 2.2. The following proofs might become clearer if we use a notation which is more familiar to the reader. We will therefore denote the equivalence class $[(m, n)]$ of (m, n) by

$$[m - n],$$

without specifying what the new symbol $(-)$ means.

Proposition 2.3. *$(\sim)$ defines an equivalence relation on $\mathbb{N} \times \mathbb{N}$.*

Proof. Reflexivity and symmetry of $(\sim)$ directly follow from the reflexivity and symmetry of equality $(=)$.

Let $(m,n),(k,\ell),(i,j) \in \mathbb{N}\times\mathbb{N}$ such that $(m,n)\sim(k,\ell)$ and $(k,\ell)\sim(i,j)$. Then $m+\ell = k+n$ and $k+j = i+\ell$. So

$$(m+\ell)+(k+j) = (k+n)+(i+\ell).$$

Hence, by (2.1), (2.2) and Theorem A.2,

$$m+j = i+n.$$

So $(m,n)\sim(i,j)$. **q. e. d.**

Definition 2.4. The *set of integers* $\mathbb{Z}$ is defined as the set of all equivalence classes under this equivalence relation:

$$\mathbb{Z} := (\mathbb{N}\times\mathbb{N})/\sim\, = \{[n-m] \mid n,m \in \mathbb{N}\}.$$

The operators of *addition* and *multiplication* as well as a *strict total order* on $\mathbb{Z}$ are defined as follows:

Definition 2.5. For $m,n,k,\ell \in \mathbb{N}$, we define

$$[m-n] +_{\mathbb{Z}} [k-\ell] := [(m+k)-(n+\ell)], \tag{2.12}$$

$$[m-n] \cdot_{\mathbb{Z}} [k-\ell] := [(m\cdot k + n\cdot\ell)-(m\cdot\ell + n\cdot k)], \tag{2.13}$$

$$[m-n] <_{\mathbb{Z}} [k-\ell] :\Longleftrightarrow m+\ell < k+n. \tag{2.14}$$

Interpreting the symbol $(-)$ with our traditional notion of subtraction, the ideas behind the definitions should all become clear.

Proposition 2.6. *The binary operations of addition* $(+_{\mathbb{Z}})$ *and multiplication* $(\cdot_{\mathbb{Z}})$ *as well as the order relation* $(<_{\mathbb{Z}})$ *on* $\mathbb{Z}$ *are well-defined.*

Proof. Let $(m,n),(k,\ell),(m',n'),(k',\ell') \in \mathbb{N}\times\mathbb{N}$ such that $(m,n)\sim(m',n')$ and $(k,\ell)\sim(k',\ell')$. Then $m+n' = m'+n$ and $k+\ell' = k'+\ell$.

By applying (2.1) and (2.2),

$$\begin{aligned}(m+k)+(n'+\ell') &= (m+n')+(k+\ell')\\ &= (m'+n)+(k'+\ell)\\ &= (m'+k')+(n+\ell).\end{aligned}$$

Hence, $(m+k, n+\ell)\sim(m'+k', n'+\ell')$.

The proof that multiplication is well-defined requires numerous simple algebraic manipulations. It is described in [4], Lemma 5ZE.

Finally, by (2.1), (2.2), (2.8) and Theorem A.3,

$$m+\ell < k+n \Longleftrightarrow m+\ell+n'+\ell' < k+n+n'+\ell'$$

$$\iff m' + n + \ell + \ell' < k' + \ell + n + n'$$
$$\iff m' + \ell' < k' + n'.$$

Hence, $[m - n] < [k - \ell]$ if and only if $[m' - n'] < [k' - \ell']$.

q. e. d.

Notation 2.7. For any integer $[m - n]$, we denote $[n - m]$ by $-[m - n]$ and call it its *negative*. Integers of the form $[n - 0]$ are referred to as $\mathbf{n}$ or $n_{\mathbb{Z}}$. The natural embedding of $\mathbb{N}$ in $\mathbb{Z}$ is given by

$$\mathbb{N}_{\mathbb{Z}} := \{[n - 0] \mid n \in \mathbb{N}\},$$

with the inclusion map $n \mapsto [n - 0] = \mathbf{n}$.

A new binary operator on $\mathbb{Z}$, *subtraction* $(-_{\mathbb{Z}})$, is defined as

$$\mathbf{x} -_{\mathbb{Z}} \mathbf{y} := \mathbf{x} + (-\mathbf{y}),$$

for any $\mathbf{x}, \mathbf{y} \in \mathbb{Z}$.

An integer $\mathbf{x}$ is called *positive* if $\mathbf{x} >_{\mathbb{Z}} 0$. It is called *negative* if $\mathbf{x} <_{\mathbb{Z}} 0$. We denote the *set of positive integers* by

$$\mathbb{Z}^+ := \{\mathbf{x} \in \mathbb{Z} \mid \mathbf{x} >_{\mathbb{Z}} 0\}.$$

Note that $\mathbb{Z}^+ = \mathbb{N}_{\mathbb{Z}} \setminus \{\mathbf{0}\}$.

Proposition 2.8. *Every integer is of the form* $\mathbf{m}$ *or* $-\mathbf{m}$ *for some* $m \in \mathbb{N}$.

Proof. Let $[k - \ell] \in \mathbb{Z}$. By Theorem A.4, there are three distinct cases.

Case 1, $k = \ell$.
Then $k + 0 = 0 + \ell$. So $[k - \ell] = [0 - 0] = \mathbf{0}$.

Case 2, $\exists m \in \mathbb{N} : k = \ell + m$.
Then $k + 0 = m + \ell$. So $[k - \ell] = [m - 0] = \mathbf{m}$.

Case 3, $\exists m \in \mathbb{N} : \ell = k + m$.
Then $k + m = 0 + \ell$. So $[k - \ell] = [0 - m] = -[m - 0] = -\mathbf{m}$. **q. e. d.**

By the previous proposition, every integer $\mathbf{x} \in \mathbb{Z}$ can be expressed as $*\mathbf{n}$ for some $n \in \mathbb{N}$, where $*$ is the *sign* of $\mathbf{x}$ (either a place holder or a minus sign).

Proposition 2.9. *For any* $m, n \in \mathbb{N}$ *and any signs* $*_1, *_2$,

$$*_1\mathbf{m} \cdot *_2\mathbf{n} = *_1 *_2 (mn)_{\mathbb{Z}}.$$

Proof. Four cases need to be checked:

Case 1.

$$\mathbf{m} \cdot_{\mathbb{Z}} \mathbf{n} = [m - 0] \cdot_{\mathbb{Z}} [n - 0] = [(mn) - 0] = (mn)_{\mathbb{Z}}.$$

Case 2.

$$-\mathbf{m} \cdot_{\mathbb{Z}} \mathbf{n} = [0 - m] \cdot_{\mathbb{Z}} [n - 0] = [0 - (mn)] = -[(mn) - 0] = -(mn)_{\mathbb{Z}}.$$

Case 3.

$$\mathbf{m} \cdot_{\mathbb{Z}} (-\mathbf{n}) = [m - 0] \cdot_{\mathbb{Z}} [0 - n] = [0 - (mn)] = -[(mn) - 0] = -(mn)_{\mathbb{Z}}.$$

Case 4.

$$(-\mathbf{m}) \cdot_{\mathbb{Z}} (-\mathbf{n}) = [0 - m] \cdot_{\mathbb{Z}} [0 - n] = [(mn) - 0] = (mn)_{\mathbb{Z}} = -(-(mn)_{\mathbb{Z}}).$$

q. e. d.

Note that two consecutive minus signs leave the number unchanged. So $*_1 *_2$ can always be substituted by some sign $*_3$.

Using the tools for the natural numbers which were given in the previous section, we can show that $(\mathbb{Z}, +_{\mathbb{Z}}, \cdot_{\mathbb{Z}}, <_{\mathbb{Z}})$ indeed satisfies the axioms for an ordered ring given in Definition A.1:

Theorem 2.10. $(\mathbb{Z}, +_{\mathbb{Z}}, \cdot_{\mathbb{Z}}, <_{\mathbb{Z}})$ *forms an ordered ring.*

Proof. Let $\mathbf{x} = [x_1 - x_2], \mathbf{y} = [y_1 - y_2], \mathbf{z} = [z_1 - z_2] \in \mathbb{Z}$.

(A1)
$$\begin{aligned}(\mathbf{x} +_{\mathbb{Z}} \mathbf{y}) +_{\mathbb{Z}} \mathbf{z} &= [((x_1 + y_1) + z_1) - ((x_2 + y_2) + z_2)] \\ &\overset{(2.1)}{=} [(x_1 + (y_1 + z_1)) - (x_2 + (y_2 + z_2))] \\ &= \mathbf{x} +_{\mathbb{Z}} (\mathbf{y} +_{\mathbb{Z}} \mathbf{z}). \end{aligned} \tag{2.15}$$

(A2)
$$\begin{aligned}\mathbf{x} +_{\mathbb{Z}} \mathbf{y} &= [(x_1 + y_1) - (x_2 + y_2)] \\ &\overset{(2.2)}{=} [(y_1 + x_1) - (y_2 + x_2)] \\ &= \mathbf{y} +_{\mathbb{Z}} \mathbf{x}. \end{aligned} \tag{2.16}$$

(A3)
$$\begin{aligned}\mathbf{x} +_{\mathbb{Z}} \mathbf{0} &= [(x_1 + 0) - (x_2 + 0)] \\ &\overset{(2.3)}{=} [x_1 - x_2] \\ &= \mathbf{x}. \end{aligned} \tag{2.17}$$

(A4)
$$\begin{aligned}\mathbf{x} +_{\mathbb{Z}} (-\mathbf{x}) &= [(x_1 + x_2) - (x_2 + x_1)] \\ &\overset{(2.2)}{=} [(x_1 + x_2) - (x_1 + x_2)] \\ &= [0 - 0] \\ &= \mathbf{0}. \end{aligned} \tag{2.18}$$

By Proposition 2.8, there are $m, n, k \in \mathbb{N}$ and signs $*_1, *_2, *_3$ such that $\mathbf{x} = *_1\mathbf{m}, \mathbf{y} = *_2\mathbf{n}, \mathbf{z} = *_3\mathbf{k}$. We can now apply Proposition 2.9.

(M1)
$$\begin{aligned}(\mathbf{x} \cdot_{\mathbb{Z}} \mathbf{y}) \cdot_{\mathbb{Z}} \mathbf{z} &= (*_1\mathbf{m} \cdot_{\mathbb{Z}} *_2\mathbf{n}) \cdot_{\mathbb{Z}} *_3\mathbf{k} \\ &= *_1 *_2 *_3((m \cdot n) \cdot k)_{\mathbb{Z}} \\ &\overset{(2.4)}{=} *_1 *_2 *_3(m \cdot (n \cdot k))_{\mathbb{Z}} \\ &= *_1\mathbf{m} \cdot_{\mathbb{Z}} (*_2\mathbf{n} \cdot_{\mathbb{Z}} *_3\mathbf{k}) \\ &= \mathbf{x} \cdot_{\mathbb{Z}} (\mathbf{y} \cdot_{\mathbb{Z}} \mathbf{z}).\end{aligned} \tag{2.19}$$

(M2)
$$\begin{aligned}\mathbf{x} \cdot_{\mathbb{Z}} \mathbf{y} &= *_1\mathbf{m} \cdot_{\mathbb{Z}} *_2\mathbf{n} \\ &= *_1 *_2 (m \cdot n)_{\mathbb{Z}} \\ &\overset{(2.5)}{=} *_1 *_2 (n \cdot m)_{\mathbb{Z}} \\ &= *_1\mathbf{n} \cdot_{\mathbb{Z}} *_2\mathbf{m} \\ &= \mathbf{y} \cdot_{\mathbb{Z}} \mathbf{x}.\end{aligned} \tag{2.20}$$

(M3)
$$\begin{aligned}\mathbf{x} \cdot_{\mathbb{Z}} \mathbf{1} &= [((x_1 \cdot 1) + (x_2 \cdot 0)) - ((x_1 \cdot 0) + (x_2 \cdot 1))] \\ &\overset{(2.6)}{=} [x_1 - x_2] \\ &= \mathbf{x}.\end{aligned} \tag{2.21}$$

(D)
$$\begin{aligned}\mathbf{x} \cdot_{\mathbb{Z}} (\mathbf{y} +_{\mathbb{Z}} \mathbf{z}) &= [x_1 - x_2] \cdot_{\mathbb{Z}} ([y_1 - y_2] +_{\mathbb{Z}} [z_1 - z_2]) \\ &= [x_1 - x_2] \cdot_{\mathbb{Z}} [(y_1 + z_1) - (y_2 + z_2)] \\ &= [(x_1(y_1 + z_1) + x_2(y_2 + z_2)) - (x_1(y_2 + z_2) + x_2(y_1 + z_1))] \\ &\overset{(2.7)}{=} [(x_1y_1 + x_1z_1 + x_2y_2 + x_2z_2) - (x_1y_2 + x_1z_2 + x_2y_1 + x_2z_1)] \\ &= ([x_1 - x_2] \cdot_{\mathbb{Z}} [y_1 - y_2]) +_{\mathbb{Z}} ([x_1 - x_2] \cdot_{\mathbb{Z}} [z_1 - z_2]) \\ &= (\mathbf{x} \cdot_{\mathbb{Z}} \mathbf{y}) +_{\mathbb{Z}} (\mathbf{x} \cdot_{\mathbb{Z}} \mathbf{z}).\end{aligned} \tag{2.22}$$

Note that for (M3) we also used the fact that $\ell \cdot 0 = 0$ for every $\ell \in \mathbb{N}$.

(O1)
$$\begin{aligned}\mathbf{x} <_{\mathbb{Z}} \mathbf{y} &\Longrightarrow x_1 + y_2 < x_2 + y_1 \\ &\overset{(2.8)}{\Longrightarrow} x_1 + z_1 + y_2 + z_2 < x_2 + z_2 + y_1 + z_1 \\ &\Longrightarrow \mathbf{x} +_{\mathbb{Z}} \mathbf{z} <_{\mathbb{Z}} \mathbf{y} +_{\mathbb{Z}} \mathbf{z}.\end{aligned} \tag{2.23}$$

(O2) Suppose that $\mathbf{z} >_{\mathbb{Z}} \mathbf{0}$. Then $\mathbf{z} = \mathbf{k} = [k - 0]$. Suppose further that $\mathbf{x} <_{\mathbb{Z}} \mathbf{y}$. Since then $x_1 + y_2 < x_2 + y_1$, we obtain by (2.9) that $k(x_1 + y_2) < k(x_2 + y_1)$. Hence,

$$\mathbf{z} \cdot_{\mathbb{Z}} \mathbf{x} = [(kx_1) - (kx_2)] <_{\mathbb{Z}} [(ky_1) - (ky_2)] = \mathbf{z} \cdot_{\mathbb{Z}} \mathbf{y}. \tag{2.24}$$

(O3) Suppose that $\mathbf{x} <_{\mathbb{Z}} \mathbf{y}$ and $\mathbf{y} <_{\mathbb{Z}} \mathbf{z}$. Since then $x_1 + y_2 < x_2 + y_1$ and $y_1 + z_2 < y_2 + z_1$, we obtain by (2.8) that $x_1 + y_1 + y_2 + z_2 < x_2 + y_1 + y_2 + z_1$. Hence, by Theorem A.3, $x_1 + z_2 < x_2 + z_1$. Thus,

$$\mathbf{x} <_{\mathbb{Z}} \mathbf{z}. \tag{2.25}$$

(O4) By (2.11), exactly one of the following three cases holds: $x_1 + y_2 < x_2 + y_1$, or $x_1 + y_2 = x_2 + y_1$, or $x_1 + y_2 > x_2 + y_1$. These correspond to:

$$\mathbf{x} <_{\mathbb{Z}} \mathbf{y}, \ \mathbf{x} = \mathbf{y}, \ \mathbf{x} >_{\mathbb{Z}} \mathbf{y}, \tag{2.26}$$

respectively. **q. e. d.**

Since we have shown that $\mathbb{Z}$ is an ordered ring and therefore satisfies all the properties of an ordered semiring, the structure of $\mathbb{N}_{\mathbb{Z}}$ is preserved. We will from now on consider $\mathbb{N}$ in the form of its embedding $\mathbb{N}_{\mathbb{Z}}$ as a subset of $\mathbb{Z}$. In particular, we can now use the same symbols for constants, operations and relations whenever we refer to naturals and integers.

The tools which we have established are enough for us to proceed to the rational numbers in a similar fashion.

2.3 From the integers to the rationals

In this section, the set of the rational numbers, denoted by $\mathbb{Q}$, will be constructed from $\mathbb{Z}$. The ideas and methods used will exhibit many similarities to the ones for the construction of $\mathbb{Z}$ from $\mathbb{N}$. We will therefore only give the relevant definitions and prove the most important properties of the newly established set $\mathbb{Q}$. Further justification can be found in [7] Section 2.4, [9] Chapter 2, and [4] Chapter 5, pp. 101 – 111.

Axiomatically, the set of rational numbers $\mathbb{Q}$ is introduced as the smallest ordered field. That is, every ordered field K contains a subfield isomorphic to $\mathbb{Q}$. This subfield is the one created by 0_K and 1_K.

The additional axiom which a field satisfies but a ring does not is the existence of a multiplicative inverse for every non-zero element.

Again, this axiomatic approach does not give us a set theoretical definition but instead postulates the existence of multiplicative inverses and integer multiples thereof. To define those new elements rigorously in a set theoretical context we need, in some sense, a notion of division.

The trick, once again, is to consider equivalence classes of ordered pairs $(m, n) \in \mathbb{Z} \times \mathbb{Z}^+$. Informally, the pair (m, n) corresponds to the rational number $\frac{m}{n}$. As in the previous section, the pair corresponding to a particular rational number is not unique. We therefore define an equivalence relation on $\mathbb{Z} \times \mathbb{Z}^+$, based on the idea that two fractions $\frac{m}{n}$ and $\frac{k}{\ell}$ are equal if and only if $m \cdot \ell = k \cdot n$.

Definition 2.11. We define a relation $(\sim)$ on $\mathbb{Z} \times \mathbb{Z}^+$ by

$$(m, n) \sim (k, \ell) :\Longleftrightarrow m \cdot \ell = k \cdot n.$$

Proposition 2.12. ($\sim$) *defines an equivalence relation.*

Notation 2.13. We denote the equivalence class $[(m, n)]$ of the pair of integers (m, n) by

$$\frac{m}{n}.$$

A formal definition of division will be given later on.

Definition 2.14. The *set of rational numbers* $\mathbb{Q}$ is defined as the set of all equivalence classes under ($\sim$):

$$\mathbb{Q} := (\mathbb{Z} \times \mathbb{Z}^+)/\sim = \left\{ \frac{m}{n} \;\middle|\; (m, n) \in \mathbb{Z} \times \mathbb{Z}^+ \right\}.$$

As is necessary for an ordered field, we need to define addition, multiplication and an order relation.

Definition 2.15. For $m, k \in \mathbb{Z}$, $n, \ell \in \mathbb{Z}^+$, we define

$$\frac{m}{n} +_{\mathbb{Q}} \frac{k}{\ell} := \frac{(m \cdot \ell) + (k \cdot n)}{n \cdot \ell}, \tag{2.27}$$

$$\frac{m}{n} \cdot_{\mathbb{Q}} \frac{k}{\ell} := \frac{m \cdot k}{n \cdot \ell}, \tag{2.28}$$

$$\frac{m}{n} <_{\mathbb{Q}} \frac{k}{\ell} :\Longleftrightarrow m \cdot \ell < k \cdot n. \tag{2.29}$$

Proposition 2.16. *The binary operations of addition ($+_{\mathbb{Q}}$) and multiplication ($\cdot_{\mathbb{Q}}$) as well as the order relation ($<_{\mathbb{Q}}$) on $\mathbb{Q}$ are well-defined.*

Notation 2.17. For any $n \in \mathbb{Z}$, we denote $\frac{n}{1}$ by $\mathbf{n}$ or $n_{\mathbb{Q}}$. The natural embedding of $\mathbb{Z}$ in $\mathbb{Q}$ is given by the map

$$n \mapsto \frac{n}{1} = \mathbf{n} = n_{\mathbb{Q}}.$$

This induces a copy of $\mathbb{Z}$ in $\mathbb{Q}$:

$$\mathbb{Z}_{\mathbb{Q}} := \left\{ \frac{n}{1} \;\middle|\; n \in \mathbb{Z} \right\}.$$

For any rational number $\mathbf{x} = \frac{m}{n}$, its *negative* is given by

$$-\mathbf{x} = -\frac{m}{n} := \frac{-m}{n}.$$

If $m \neq 0$, we define

$$\left(\frac{m}{n}\right)^{-1} := \begin{cases} \frac{n}{m} & \text{if } m > 0, \\ \frac{-n}{-m} & \text{if } m < 0. \end{cases}$$

The binary operator subtraction ($-_{\mathbb{Q}}$) is, as before, defined as

$$\mathbf{x} -_{\mathbb{Q}} \mathbf{y} = \mathbf{x} + (-\mathbf{y}),$$

for any $\mathbf{x}, \mathbf{y} \in \mathbb{Q}$.

A new binary operator $(/_{\mathbb{Q}})$, called *division*, on $\mathbb{Q} \times \mathbb{Q} \setminus \{\mathbf{0}\}$ is defined by

$$\frac{\mathbf{x}}{\mathbf{y}} = \mathbf{x}/_{\mathbb{Q}}\mathbf{y} := \mathbf{x} \cdot \mathbf{y}^{-1}.$$

We can now proceed to showing that $(\mathbb{Q}, +_{\mathbb{Q}}, \cdot_{\mathbb{Q}}, <_{\mathbb{Q}})$ with additive identity $\mathbf{0}$ and multiplicative identity $\mathbf{1}$ is an ordered field using the properties (2.15) – (2.26) of the ordered ring $\mathbb{Z}$. We will, however, only prove the new property of $\mathbb{Q}$ — the existence of multiplicative inverses for non-zero elements. Detailed proofs of the other properties can be found, for example, in [4], p. 104–109.

Proposition 2.18. $(\mathbb{Q}, +_{\mathbb{Q}}, \cdot_{\mathbb{Q}}, <_{\mathbb{Q}})$ *forms an ordered field.*

Proof. (M4) Let $\mathbf{x} = \frac{m}{n} \in \mathbb{Q} \setminus \{\mathbf{0}\}$. Suppose that $m > 0$. Then

$$\begin{aligned} \mathbf{x} \cdot_{\mathbb{Q}} \mathbf{x}^{-1} &= \left(\frac{m}{n}\right) \cdot_{\mathbb{Q}} \left(\frac{n}{m}\right) \\ &= \frac{m \cdot n}{n \cdot m} \\ &\overset{(2.20)}{=} \frac{m \cdot n}{m \cdot n}. \end{aligned}$$

Since $(m \cdot n) \cdot 1 = m \cdot n = 1 \cdot (m \cdot n)$, we obtain

$$\mathbf{x} \cdot_{\mathbb{Q}} \mathbf{x}^{-1} = \frac{m \cdot n}{m \cdot n} = \frac{1}{1} = \mathbf{1}. \tag{2.30}$$

If $n < 0$, we have $\mathbf{x}^{-1} = \frac{-n}{-m}$. In that case we can argue similarly to obtain

$$\mathbf{x} \cdot_{\mathbb{Q}} \mathbf{x}^{-1} = \frac{-(m \cdot n)}{-(m \cdot n)} = \mathbf{1}.$$

q. e. d.

Since $\mathbb{Q}$ satisfies the axioms for an ordered field, which entirely includes the axioms for an ordered ring, we can identify $\mathbb{Z}$ with the subset $\mathbb{Z}_{\mathbb{Q}}$ of $\mathbb{Q}$ and use the same symbols for operators and relations on those sets.

Establishing the ordered field of rational numbers is the prerequisite for the actual set we are aiming for: the complete ordered field of real numbers. We still need to give a formal description of completeness as well as methods of filling the "gaps" in $\mathbb{Q}$, which we will explore in the next chapter.

3 Classical approaches to completeness

Already in the ancient world it was known that there exist numbers representing a length in geometry which is not representable as a rational number. These *'irrational' numbers* are the "gaps" on the continuous number line which are not filled by rationals. This already gives us an intuitive concept of completeness: The real number line is complete means that there are no such gaps which are not covered by a real number.

As it turns out, the complete ordered field of real numbers $\mathbb{R}$ is unique up to isomorphism (see section 5.2); in fact, there is a unique isomorphism from one complete ordered field to another. There are various equivalent ways of formally describing the completeness property. For instance, one common notion of completeness, which was described in the introduction, is the supremum property. Depending on how the problem of "filling the gaps" is approached, different but equivalent notions of completeness naturally arise.

In the following two sections, two classical approaches will be demonstrated in detail: Dedekind's construction through cuts on $\mathbb{Q}$ and Cantor's construction through rational Cauchy sequences.

Dedekind cuts and resulting properties are fully described in [9] Chapter 3 & 4, and [7] Section 2.2. Section 2.3 in [7] also presents Cantor's approach through Cauchy sequences. The following two sections are largely inspired by those parts of the books.

3.1 Dedekind's construction through cuts

First we try to approach completeness by cuts on the set of rational numbers. Landau defines in [9], p. 43, a cut as a set of rational numbers satisfying the following three conditions:

1. It contains a rational number, but it does not contain all rational numbers.
2. Every rational number of the set is smaller than every rational number not contained in the set.
3. It does not contain a greatest rational number.

We formalise this as follows:

Definition 3.1 (*Dedekind cuts*)**.** A set $\mathbf{r} \subset \mathbb{Q}$ is called a *(Dedekind) cut* (on $\mathbb{Q}$) if it satisfies the following three conditions:

$$\emptyset \neq \mathbf{r} \neq \mathbb{Q} \tag{3.1}$$

$$\forall p \in \mathbf{r}\ \forall q \in \mathbb{Q} \setminus \mathbf{r} :\ p < q \tag{3.2}$$

$$\forall p \in \mathbf{r}\ \exists q \in \mathbf{r}:\ p < q. \tag{3.3}$$

We now define the set of real numbers $\mathbb{R}$ as the set of all Dedekind cuts on $\mathbb{Q}$. Once we have introduced other constructions of the real numbers, we will refer to these as Dedekind real numbers and call them $\mathbb{R}_\mathrm{D}$.

Using the usual notion of suprema, the set $\mathbf{r}$ does indeed represent the corresponding number $\sup \mathbf{r}$ in $\mathbb{R}$. The idea behind the corresponding version of completeness is that every Dedekind cut has a least upper bound in $\mathbb{R}$. This is in fact already one form of the completeness axiom, which will be further discussed later in this section.

Notation 3.2. For $q \in \mathbb{Q}$, we denote $\{x \in \mathbb{Q} \mid x < q\}$ by $\mathbf{q}$ or $q_\mathbb{R}$.

The natural copy of $\mathbb{Q}$ in $\mathbb{R}$ is given by

$$\mathbb{Q}_\mathbb{R} := \{\mathbf{q} \mid q \in \mathbb{Q}\},$$

with the natural embedding map $q \mapsto \mathbf{q}$.

Note that for any $p \in \mathbb{Q}$, we have $p \in \mathbf{q}$ if and only if $p < q$.

It is easy to see that $\mathbf{q}$ defines a Dedekind cut whose least upper bound is q. Once we define the necessary operators and order relation, one can show that $\mathbb{Q}_\mathbb{R}$ is indeed an ordered field. For a detailed proof, see [4] Theorem 5RJ.

We proceed by defining addition ($+_\mathbb{R}$), multiplication ($\cdot_\mathbb{R}$) and an order relation ($<_\mathbb{R}$) on $\mathbb{R}$.

Definition 3.3. Let $\mathbf{r}, \mathbf{s} \in \mathbb{R}$.

$$\mathbf{r} +_\mathbb{R} \mathbf{s} := \{p + q \mid p \in \mathbf{r}, q \in \mathbf{s}\}, \tag{3.4}$$

$$\mathbf{r} <_\mathbb{R} \mathbf{s} :\Longleftrightarrow \mathbf{r} \subsetneq \mathbf{s}. \tag{3.5}$$

Undoubtedly the order relation $<_\mathbb{R}$ is well-defined. It will be useful to show that trichotomy holds.

Proposition 3.4. *For any $\mathbf{r}, \mathbf{s} \in \mathbb{R}$, exactly one of the following holds:*

$$\mathbf{r} <_\mathbb{R} \mathbf{s}, \quad \mathbf{r} = \mathbf{s}, \quad \mathbf{r} >_\mathbb{R} \mathbf{s}.$$

Proof. Let $\mathbf{r}, \mathbf{s} \in \mathbb{R}$ such that $\mathbf{r} \neq \mathbf{s}$. Then either $\mathbf{r} \setminus \mathbf{s} \neq \emptyset$ or $\mathbf{s} \setminus \mathbf{r} \neq \emptyset$. Suppose that $\mathbf{r} \setminus \mathbf{s} \neq \emptyset$ and take $q \in \mathbf{r} \setminus \mathbf{s}$. Since $q \in \mathbf{r}$, all rationals less than q are also contained in $\mathbf{r}$. As $q \notin \mathbf{s}$, all rationals in $\mathbf{s}$ are smaller than q. Hence, all elements of $\mathbf{s}$ are contained in $\mathbf{r}$. So $\mathbf{s} \subsetneq \mathbf{r}$, whence $\mathbf{s} <_\mathbb{R} \mathbf{r}$.

We can argue similarly to show that if $\mathbf{s} \setminus \mathbf{r} \neq \emptyset$, then $\mathbf{r} <_\mathbb{R} \mathbf{s}$. **q. e. d.**

For the definition of multiplication on the real numbers we have to be more careful. We will first define the product of non-negative real numbers.

Definition 3.5. Let $\mathbf{r}, \mathbf{s} \in \mathbb{R}$ such that $\mathbf{r}, \mathbf{s} \geq_{\mathbb{R}} \mathbf{0}$. Then

$$\mathbf{r} \cdot_{\mathbb{R}} \mathbf{s} := \{p \cdot q \mid p \in \mathbf{r} \setminus \mathbf{0}, q \in \mathbf{s} \setminus \mathbf{0}\} \cup \mathbf{0}. \tag{3.6}$$

We must exclude negative numbers from the sets $\mathbf{r}$ and $\mathbf{s}$, so that the products of the rationals they contain do not become arbitrarily large. As similar problems are encountered for the definition of products of other real numbers, it will be useful to define the negative and the modulus of a real number first.

Definition 3.6. Let $\mathbf{r} \in \mathbb{R}$. The *negative* of $\mathbf{r}$ is defined as

$$-\mathbf{r} := \{q \in \mathbb{Q} \mid \exists p > q : -p \in \mathbb{Q} \setminus \mathbf{r}\}.$$

Proposition 3.7. *For any $\mathbf{r} \in \mathbb{R}$, its negative $-\mathbf{r}$ is a Dedekind cut.*

Proof. Let $q \in \mathbf{r}$ and $s \in \mathbb{Q}$ such that $s > -q$. Then $-s < q$, so $-s \in \mathbf{r}$. Hence, $-q \notin -\mathbf{r}$. Now let $u \in \mathbb{Q} \setminus \mathbf{r}$. Since $-u > -u - 1$ and $-(-u) = u \in \mathbb{Q} \setminus \mathbf{r}$, the rational $-u - 1$ lies in $-\mathbf{r}$.

Suppose that q' is a rational less than q. Since $q \in \mathbf{r}$, there exists $p > q > q'$ such that $-p \in \mathbb{Q} \setminus \mathbf{r}$. Thus, $q' \in -\mathbf{r}$.

Next we need to show that $-\mathbf{r}$ contains no greatest element. Let $t := \frac{q+p}{2}$. Then $q < t < p$. As t is rational and $p \in \mathbb{Q} \setminus \mathbf{r}$, it is also contained in $-\mathbf{r}$. So for every element in $-\mathbf{r}$, we can find a greater element also lying in $-\mathbf{r}$. **q. e. d.**

Definition 3.8. The *modulus function* from $\mathbb{R}$ to the set of non-negative real numbers $\mathbb{R}_{\geq 0}$ is defined as

$$|\ .\ | : \mathbb{R} \to \mathbb{R}_{\geq 0}, \mathbf{r} \mapsto \begin{cases} \mathbf{r} & \text{if } \mathbf{r} \geq_{\mathbb{R}} \mathbf{0}, \\ -\mathbf{r} & \text{if } \mathbf{r} <_{\mathbb{R}} \mathbf{0}. \end{cases}$$

$|\mathbf{r}|$ is called the *absolute value* of $\mathbf{r}$.

An equivalent, rather set theoretical definition of the modulus would be $|\mathbf{r}| = \mathbf{r} \cup -\mathbf{r}$.

Now we can define multiplication of two real numbers in the remaining cases.

Definition 3.9. Let $\mathbf{r}, \mathbf{s} \in \mathbb{R}$.

$$\mathbf{r} \cdot_{\mathbb{R}} \mathbf{s} := \begin{cases} -(\mathbf{r} \cdot_{\mathbb{R}} |\mathbf{s}|) & \text{if } \mathbf{r} \geq_{\mathbb{R}} \mathbf{0}, \mathbf{s} <_{\mathbb{R}} \mathbf{0}, \\ -(|\mathbf{r}| \cdot_{\mathbb{R}} \mathbf{s}) & \text{if } \mathbf{r} <_{\mathbb{R}} \mathbf{0}, \mathbf{s} \geq_{\mathbb{R}} \mathbf{0}, \\ |\mathbf{r}| \cdot_{\mathbb{R}} |\mathbf{s}| & \text{if } \mathbf{r}, \mathbf{s} <_{\mathbb{R}} \mathbf{0}. \end{cases}$$

Unlike in the previous section, it is not obvious that the real numbers are closed under addition and multiplication. We need to check whether the sum and product of two reals are contained in $\mathbb{R}$.

Proposition 3.10. $\mathbb{R}$ *is closed under addition* ($+_{\mathbb{R}}$) *and multiplication* ($\cdot_{\mathbb{R}}$).

Proof. Let $\mathbf{r}, \mathbf{s} \in \mathbb{R}$.

(1) Let $q \in \mathbf{r}$, $a \in \mathbb{Q} \setminus \mathbf{r}$, $p \in \mathbf{s}$ and $b \in \mathbb{Q} \setminus \mathbf{s}$. Then $q + p \in \mathbf{r} + \mathbf{s}$. For any $q' \in \mathbf{r}$ and $p' \in \mathbf{s}$, we have $q' < a$ and $p' < b$. So $q' + p' < a + b$. Thus, $a + b \notin \mathbf{r} + \mathbf{s}$.

Now let $c \in \mathbb{Q} \setminus (\mathbf{r} + \mathbf{s})$. Since $c \neq q + p$, either $c < q + p$ or $c > q + p$. But if $c < q + p$, then $c - q < p$. So $c - q \in \mathbf{s}$, and thus $c \in \mathbf{r} + \mathbf{s}$, a contradiction. Hence, $c > q + p$.

Finally, we need to show that $\mathbf{q} + \mathbf{s}$ contains no greatest element. Let $u \in \mathbf{r}$ and $v \in \mathbf{s}$ such that $q < u$ and $p < v$. Then $p + q < u + v \in \mathbf{r} +_{\mathbb{R}} \mathbf{s}$.

(2) Suppose that $\mathbf{r}, \mathbf{s} >_{\mathbb{R}} \mathbf{0}$. Let $q \in \mathbf{r} \setminus \mathbf{0}$ and $p \in \mathbf{s} \setminus \mathbf{0}$. The product $\mathbf{r} \cdot_{\mathbb{R}} \mathbf{s}$ is nonempty, as $\mathbf{0} \subseteq \mathbf{r} \cdot_{\mathbb{R}} \mathbf{s}$. Let $a \in \mathbb{Q} \setminus \mathbf{r}$ and $b \in \mathbb{Q} \setminus \mathbf{s}$. Then $0 < q < a$ and $0 < p < b$. So $q \cdot p < a \cdot b$, whence $a \cdot b \notin \mathbf{r} \cdot_{\mathbb{R}} \mathbf{s}$.

Let $c \in \mathbb{Q}$ such that $c < q \cdot p$. If $c \in \mathbf{0}$, then $c \in \mathbf{r} \cdot_{\mathbb{R}} \mathbf{s}$ already. If $c \notin \mathbf{0}$, then $c \geq 0$. In particular then $q > 0$, and $0 \leq c \cdot q^{-1} < p$. So $c \cdot q^{-1} \in \mathbf{s} \setminus \mathbf{0}$. Thus, $c \in \mathbf{r} \cdot_{\mathbb{R}} \mathbf{s}$.

Finally, for $u \in \mathbf{r}$ and $v \in \mathbf{s}$ such that $q < u$ and $p < v$, since $0 < u$ and $0 < v$, we obtain $q \cdot p < u \cdot v \in \mathbf{r} \cdot_{\mathbb{R}} \mathbf{s}$.

If $\mathbf{r} = \mathbf{0}$ or $\mathbf{s} = \mathbf{0}$, their product in $\mathbb{R}$ is $\mathbf{0}$.

In the remaining cases the product is a Dedekind cut by definition. **q. e. d.**

To show that $\mathbb{R}$ is a field, we need to introduce multiplicative inverses.

Definition 3.11. For any $\mathbf{s} \in \mathbb{R} \setminus \{\mathbf{0}\}$, we define its *multiplicative inverse* as follows: If $\mathbf{s} >_{\mathbb{R}} \mathbf{0}$, then

$$\mathbf{s}^{-1} := \left\{ q \in \mathbb{Q} \setminus \{0\} \mid \exists p \in \mathbb{Q} \setminus \mathbf{s} : p < q^{-1} \right\} \cup \mathbf{0} \cup \{0\}.$$

If $\mathbf{s} <_{\mathbb{R}} \mathbf{0}$, then

$$\mathbf{s}^{-1} := -|\mathbf{s}|^{-1}.$$

For a proof that $\mathbf{s}^{-1}$ defines a cut, see [9] Theorem 152.

Theorem 3.12. $(\mathbb{R}, +_{\mathbb{R}}, \cdot_{\mathbb{R}}, <_{\mathbb{R}})$ *is an ordered field.*

Proof. Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}$.

Properties (A1)–(A2) follow from accociativity and commutativity of $\mathbb{Q}$ under addition.

(A3) To show: $\mathbf{x} + \mathbf{0} = \mathbf{0}$.

Let $q \in \mathbf{x}$ and $p \in \mathbf{0}$. Since $p < 0$, we obtain $q + p < q$. So $q + p \in \mathbf{x}$. Hence, $\mathbf{x} +_{\mathbb{R}} \mathbf{0} \subseteq \mathbf{x}$.

Now suppose that $t \in \mathbf{x}$. Let $t' \in \mathbf{x}$ such that $t < t'$. Then $t - t' \in \mathbf{0}$. Hence, $t = t' + (t - t') \in \mathbf{x} + \mathbf{0}$. Hence, $\mathbf{x} \subseteq \mathbf{x} +_{\mathbb{R}} \mathbf{0}$.

(A4) See [4] Theorem 5RF.

(M1)–(M3) These can first be demonstrated for positive real numbers. We can then use the identity $-(-\mathbf{x}) = \mathbf{x}$, which follows from (A2)–(A4), for the other cases.

(M4) First consider $\mathbf{x} >_{\mathbb{R}} \mathbf{0}$. Let $u \in \mathbf{x} \setminus \mathbf{0}$ and $v \in \mathbf{x}^{-1} \setminus \mathbf{0}$. Then $v^{-1} \in \mathbb{Q} \setminus \mathbf{x}$ and $u^{-1} \in \mathbb{Q} \setminus \mathbf{x}^{-1}$. If $\mathbf{x} <_{\mathbb{R}} \mathbf{x}^{-1}$, then $v < u^{-1}$. If $\mathbf{x}^{-1} <_{\mathbb{R}} \mathbf{x}$, then $u < v^{-1}$. Hence, $uv < 1$. So $\mathbf{x} \cdot_{\mathbb{R}} \mathbf{x}^{-1} \subseteq \mathbf{1}$.

The other direction, i. e. showing that $\mathbf{1} \subseteq \mathbf{x} \cdot_{\mathbb{R}} \mathbf{x}^{-1}$, is fully demonstrated in [9], Theorem 152.

For $\mathbf{x} <_{\mathbb{R}} \mathbf{0}$, the proof follows from the definition of multiplicative inverses of negative numbers.

(D) This follows immediately from distributivity in $\mathbb{Q}$.

Conditions (O1)–(O3) all follow immediately from the properties of the proper subset relation ($\subsetneq$). Trichotomy (O4) was already proved. **q. e. d.**

As $\mathbb{R}$ is an ordered field, and the structure of $\mathbb{Q}$ in $\mathbb{R}$ considered as its natural copy $\mathbb{Q}_{\mathbb{R}}$ is preserved, we can from now on use the same notation.

The important additional property of completeness, which distinguishes $\mathbb{R}$ from $\mathbb{Q}$, remains to be introduced and explained.

In [2], p. 322, Dedekind himself gives a notion of completeness of the "straight line" (meaning the real number line) which is closely related to Dedekind cuts:

"If all points on the straight line fall into two classes, such that every point of the first class lies to the left of every point of the second class, then there exists one and only one point which produces this division of all points into two classes, this dissection of the line into two pieces."

Notably, this notion of completeness uses Dedekind cuts on the real numbers and expresses a one-to-one correspondence between those cuts and the real numbers. We will formalise it and express this version of completeness for general ordered fields in the following.

Definition 3.13 (*Dedekind completeness*)**.** An ordered field F is *Dedekind complete* if *every Dedekind cut on F has a least upper bound in F.*

Remark 3.14. Dedekind cuts on $\mathbb{R}$ or any other ordered field are defined similarly to the ones on $\mathbb{Q}$.

Theorem 3.15. *The Dedekind real number system is Dedekind complete.*

Proof. Let $\mathbf{A}$ be a Dedekind cut on $\mathbb{R}$. We will first show that $\mathbf{a} := \bigcup \mathbf{A}$ is a Dedekind cut on $\mathbb{Q}$ and thus a real number, and then that $\mathbf{a}$ is the least upper bound of $\mathbf{A}$.

Since $\mathbf{A}$ is non-empty and every element of $\mathbf{A}$ is nonempty, $\mathbf{a}$ is also non-empty. Let $\mathbf{b} \in \mathbb{R} \setminus \mathbf{A}$. Then for every $\mathbf{c} \in \mathbf{A}$, we have $\mathbf{c} <_{\mathbb{R}} \mathbf{b}$. Thus, $\mathbf{c} \subsetneq \mathbf{b}$. In particular, $\mathbf{a} = \bigcup \mathbf{A} \subseteq \mathbf{b}$. Hence, $\mathbf{a} \subsetneq \mathbf{b} +_{\mathbb{R}} \mathbf{1}$. So $\mathbb{Q} \setminus \mathbf{a} \neq \emptyset$.

Let $q \in \mathbf{a}$ and $p \in \mathbb{Q} \setminus \mathbf{a} = \bigcap_{\mathbf{c} \in \mathbf{A}} (\mathbb{Q} \setminus \mathbf{c})$. Consider $\mathbf{p} = p_{\mathbb{R}} \in \mathbb{R}$. Since $p \notin \mathbf{c}$ for all $\mathbf{c} \in \mathbf{A}$, it follows that $\mathbf{c} \subsetneq \mathbf{p}$ for all $\mathbf{c} \in \mathbf{A}$. Thus, $\mathbf{a} \subseteq \mathbf{p}$. So $q \in \mathbf{p}$, and hence $q < p$.

Suppose that $s \in \mathbb{Q}$ is an upper bound for $\mathbf{a}$. Since $\mathbf{c} \subseteq \mathbf{a}$ for all $\mathbf{c} \in \mathbf{A}$, that s is also an upper bound for each $\mathbf{c}$ in $\mathbf{A}$. So s is not contained in any $\mathbf{c}$ and hence not in $\mathbf{a}$.

Hence, $\mathbf{a}$ is a Dedekind cut on $\mathbb{Q}$.

By construction, $\mathbf{c} \subseteq \mathbf{a}$, so $\mathbf{c} \leq_{\mathbb{R}} \mathbf{a}$ for all $\mathbf{c} \in \mathbf{A}$. Assume for a contradiction that there were an upper bound $\mathbf{a}' \in \mathbb{R}$ for $\mathbf{A}$ such that $\mathbf{a}' < \mathbf{a}$. Then $\mathbf{a}' \subsetneq \mathbf{a}$. For $t \in \mathbf{a} \setminus \mathbf{a}'$, we obtain $\mathbf{a}' \subsetneq \mathbf{t} \subsetneq \mathbf{a}$ and $\mathbf{t} \in \mathbf{A}$. But then $\mathbf{a}' <_{\mathbb{R}} \mathbf{t}$, contradicting that $\mathbf{a}'$ is an upper bound.

Hence, $\mathbf{a}$ is the least upper bound of A.

q. e. d.

Dedekind's notion of completeness is closely related to the version of the completeness axiom in the form of the supremum property. In fact, those two versions (as all other versions) are equivalent (see Section 5.1).

We will show that the Dedekind real number system satisfies this axiom.

Theorem 3.16. *The Dedekind real number system has the supremum property.*

Proof. Let A be a non-empty set of real numbers which is bounded above. Define the set B as follows:

$$B := \{x \in \mathbb{R} \mid \exists y \in A : x < y\}.$$

First we will show that B is a Dedekind cut on $\mathbb{R}$, and then that the least upper bound of B is the least upper bound of A.

As A is non-empty, there exists $a \in A$. Then $a - 1 \in B$. As A is bounded above, there exists $m \in \mathbb{R}$ such that $z \leq m$ for all $z \in A$. So $m \notin B$.

Let $b \in B$ and $c \in \mathbb{R} \setminus B$. Note that then c is an upper bound for B. Moreover, there exists $y \in A$ such that $b < y$. So $b < y \leq c$.

Let $b' = \frac{b+y}{2}$. Then $b < b' < y$. Hence, B has no maximum. Thus, B is a Dedekind cut on $\mathbb{R}$.

By the previous theorem, B has a least upper bound s, say. If it were not an upper bound for A, there would be $y \in A$ such that $s < y$. But then, for $b' = \frac{s+y}{2}$, we would have $s < b' < y$. This is a contradiction, since $b' \in B$. Hence, s is an upper bound for A.

Assume for a contradiction that A has an upper bound s' such that $s' < s$. Since s is the least upper bound of B, which is not contained in B, there exists $b \in B$ such that $s' < b < s$. But then there exists $a' \in A$ such that $s' < b < a'$, a contradiction.

Hence, s is the least upper bound of A. **q. e. d.**

This finishes one possible construction of the complete ordered field of real numbers. Another classical approach leading to a different version of the completeness axiom will be demonstrated in the following section.

3.2 Cantor's construction through Cauchy sequences

When we define the real numbers axiomatically and consider the set of rational numbers as a subset, we know from the study of sequences that every real Cauchy sequence converges to a real number. We also know that for every real number p, there exists a convergent sequence and hence Cauchy sequence of rational numbers converging to p. This gives us a correspondence between rational Cauchy sequences and real numbers, namely that a rational Cauchy sequence (a_n) corresponds to its limit $\lim_{n\to\infty} a_n$. We will formalise this concept in this section. Throughout the section we use the standard notation for sequences and only prove results which are not usually covered in a first year Analysis course (e.g. Earl [3]).

Let C be the set of all Cauchy sequences in $\mathbb{Q}$. As we want to define the real numbers as limits of rational Cauchy sequences, we need to make sure that two distinct Cauchy sequences with the same limit do not correspond to different real numbers. As in the previous section, the solution is to define an equivalence relation.

Definition 3.17. We define a relation $(\sim)$ on C by

$$(a_n) \sim (b_n) :\Longleftrightarrow \lim_{n\to\infty} (a_n - b_n) = 0.$$

This uses the idea that two convergent sequences tend to the same limit if and only if their difference tends to zero.

We could proceed by proving that this relation is indeed an equivalence relation and then define the necessary operators and order relation, which is fully demonstrated in [7] Section 2.3. However, knowing that C with standard addition and multiplication of sequences forms a ring, we will rephrase the equivalence relation in the context of rings and ideals and exploit results from Algebra.

Definition 3.18. We define the subset I of C as the set of zero-sequences in C:

$$I := \left\{ (a_n) \in C \;\middle|\; \lim_{n\to\infty} a_n = 0 \right\}.$$

Proposition 3.19. *I is an ideal of C.*

Proof. We need to show that I is closed under addition, and under multiplication by elements in C.

Let $(a_n), (b_n) \in I$ and $(c_n) \in C$. Since (c_n) is bounded (see Theorem A.5), $\lim_{n\to\infty} c_n a_n = 0$. Hence, $(a_n) \cdot_C (c_n) \in I$. Moreover, since (a_n) and (b_n) both converge to 0, so does their sum. Hence, $(a_n) +_C (b_n) \in I$. **q. e. d.**

As $I \triangleleft C$, the quotient C/I forms a ring with addition and multiplication induced by C. In order to show that C/I forms a field, it suffices to prove the existence of multiplicative inverses of elements not contained in I. This is equivalent to showing that I is a maximal ideal.

Proposition 3.20. *The quotient ring C/I is a field.*

Proof. Let $(d_n) \in C \setminus I$.

Define a sequence (d'_n) by

$$d'_n := \begin{cases} d_n^{-1} & \text{if } d_n \neq 0, \\ 0 & \text{if } d_n = 0. \end{cases}$$

Since (d_n) is not a zero-sequence, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $d_n \neq 0$, and the N-th tail (d'_{N+n}) of (d'_n) is equal to (d^{-1}_{N+n}). Thus, by Theorem A.6, (d^{-1}_{N+n}) is a Cauchy sequence. Hence, also (d'_n) is a Cauchy sequence. Moreover, we obtain for all $n \geq N$ that $d'_n d_n = 1$. So $(d_n d'_n - 1)$ is a zero-sequence. Hence, $(d_n) \cdot_C (d'_n) + I = 1_C + I$. **q. e. d.**

Definition 3.21. We define the field of *real numbers* $\mathbb{R}$ as the quotient of the ring of rational Cauchy sequences C and its ideal of zero-sequences I:

$$\mathbb{R} := C/I.$$

Addition $(+_\mathbb{R})$ and multiplication $(\cdot_\mathbb{R})$ are induced by C.

We also call $\mathbb{R}$ the *Cantor real number system* and denote it by $\mathbb{R}_\mathrm{C}$ if ambiguity might arise.

Remark 3.22. For any $(a_n), (b_n) \in C$,

$$(a_n) \sim (b_n) \text{ if and only if } (a_n) -_C (b_n) \in I.$$

This follows directly from the definition of I and $(\sim)$. As a result,

$$(a_n) \sim (b_n) \text{ if and only if } (a_n) + I = (b_n) + I.$$

Hence,

$$[(a_n)] = (a_n) + I \text{ and } (C/\sim) = C/I.$$

It therefore makes no difference whether we define the real numbers as the set of equivalence classes of rational Cauchy sequences or as the quotient ring of the ring of rational Cauchy sequences with its maximal ideal of zero-sequences.

Notation 3.23. For any rational number $q \in \mathbb{Q}$, the natural copy of q in $\mathbb{R}$ is given by $(q_n) + I$, where (q_n) is the constant sequence $q_n = q$ for all $n \in \mathbb{N}$.

The notation for additive and multiplicative inverses as well as subtraction and division are defined in the same way as on $\mathbb{Q}$.

Also note that $\mathbf{0} = I$.

So far we have not mentioned how the order on $\mathbb{Q}$ can be extended to an order on $\mathbb{R}$.

Definition 3.24. We define a relation $(<_R)$ on $C/\sim$ as follows:

$$[(a_n)] <_{\mathbb{R}} [(b_n)] :\Longleftrightarrow (\exists \delta \in \mathbb{Q}_{>0}\ \exists N \in \mathbb{N}\ \forall n \geq N : a_n + \delta < b_n).$$

Proposition 3.25. *The relation $(<_{\mathbb{R}})$ is well defined.*

Proof. Let $(a_n), (b_n), (a'_n), (b'_n) \in C$. Suppose that there exist $N_1 \in \mathbb{N}$ and $\delta \in \mathbb{Q}$ such that $\delta > 0$ and for all $n \geq N_1$ we have $a_n > b_n + \delta$. Suppose further that $(a_n) \sim (a'_n)$, $(b_n) \sim (b'_n)$. Then there exists $N_2 \in \mathbb{N}$ such that $a'_n > a_n - \frac{1}{3}\delta$ for all $n \geq N_2$, and there exists $N_3 \in \mathbb{N}$ such that $b_n + \frac{1}{3}\delta > b'_n$ for all $n \geq N_3$.

Let $N := \max\{N_1, N_2, N_3\}$. Then for all $n \geq N$:

$$a'_n > a_n - \frac{1}{3}\delta > b_n + \frac{2}{3}\delta > b'_n + \frac{1}{3}\delta.$$

This yields that $(<_{\mathbb{R}})$ is well-defined.

q. e. d.

Theorem 3.26. $(\mathbb{R}, +_{\mathbb{R}}, \cdot_{\mathbb{R}}, <_{\mathbb{R}})$ *forms an ordered field.*

Proof. Let $\mathbf{a} = [(a_n)], \mathbf{b} = [(b_n)], \mathbf{c} = [(c_n)] \in \mathbb{R}$. Conditions (O1) and (O2) follow directly from the properties of $\mathbb{Q}$ as an ordered field.

Property (O3) follows from (O1) and the fact that $\mathbf{a} <_{\mathbb{R}} \mathbf{b}$ if and only if $\mathbf{0} <_{\mathbb{R}} \mathbf{b} - \mathbf{a}$.

(O4) Suppose that $\mathbf{a} \neq \mathbf{b}$. Then $(a_n - b_n)$ does not tend to 0. Hence, there exists $\delta > 0$ such that for all $N \in \mathbb{N}$ there exists $n \geq N$ such that $|a_n - b_n| \geq \delta$. Since

(a_n) and (b_n) are Cauchy sequences, there exist $N_1, N_2 \in \mathbb{N}$ such that $|a_n - a_m| < \frac{1}{3}\delta$ for all $n, m \geq N_1$ and $|b_n - b_m| < \frac{1}{3}\delta$ for all $n, m \geq N_2$. Let $N := \max\{N_1, N_2\}$. Then there exists $n \geq N$ such that $(a_n - b_n) \geq \delta$ or $(a_n - b_n) \leq -\delta$. In the first case, for all $m \geq N$,

$$a_m > a_n - \frac{1}{3}\delta \geq b_n + \frac{2}{3}\delta > b_m + \frac{1}{3}\delta.$$

Thus, $\mathbf{a} >_{\mathbb{R}} \mathbf{b}$.

In the second case, for all $m \geq N$,

$$a_m < a_n + \frac{1}{3}\delta \leq b_n - \frac{2}{3}\delta < b_m + \frac{1}{3}\delta.$$

Thus, $\mathbf{a} <_{\mathbb{R}} \mathbf{b}$. **q. e. d.**

Now that we have shown that $\mathbb{R}$ is an ordered field, we can consider $\mathbb{Q}$ in the form of its natural embedding as a subset of $\mathbb{R}$ and adopt the notation on $\mathbb{Q}$. The only property left to describe is completeness.

Definition 3.27 (*Cauchy completeness*)**.** An ordered field F is *Cauchy complete* if *every Cauchy sequence in F converges to a unique limit in F.*

Remark 3.28. If F is a metric space, such as $\mathbb{R}$ with metric induced by the modulus function $(|\ .\ |)$, then the limit of a convergent sequence is automatically unique.

Theorem 3.29. *The Cantor real number system $\mathbb{R}$ is Cauchy complete.*

Proof. Let $(\mathbf{a}_n)_n$ be a Cauchy sequence in $\mathbb{R}$. For each $n, m \in \mathbb{N}$, let

$$\left(a_m^{(n)}\right)_m$$

be a representative of the equivalence class $\mathbf{a}_n \in \mathbb{R}$. That is, $\left(a_m^{(n)}\right)_m$ is a Cauchy sequence in $\mathbb{Q}$ such that

$$\mathbf{a}_n = \left[\left(a_m^{(n)}\right)_m\right].$$

Fix $n \in \mathbb{N}$. Since $\left(a_m^{(n)}\right)_m$ is a rational Cauchy sequence, there exists ℓ_n such that for all $m, m' \geq \ell_n$,

$$\left|a_m^{(n)} - a_{m'}^{(n)}\right| < \frac{1}{n}. \tag{3.7}$$

Without loss of generality, we can assume that the sequence (ℓ_n) is strictly increasing, as each ℓ_n can be chosen arbitrarily large. Now construct a new sequence in $\mathbb{Q}$ by

$$(b_n)_n := \left(a_{\ell_n}^{(n)}\right)_n.$$

We want to show that $\mathbf{b} := [(b_n)_n]$ is a real number and the limit of $(\mathbf{a}_n)$.

First, we need to show that $(b_n)_n$ is a Cauchy sequence. Let $\varepsilon \in \mathbb{Q}_{>0}$ and $\varepsilon_{\mathbb{R}}$ the corresponding real number. Choose $N \in \mathbb{N}$ such that for all $n, n' \geq N$,

$$|\mathbf{a}_n - \mathbf{a}_{n'}| <_{\mathbb{R}} \frac{\varepsilon_{\mathbb{R}}}{\mathbf{2}}.$$

This implies that there exists $M \in \mathbb{N}$ such that for all $m \geq M$,

$$\left|a_m^{(n)} - a_m^{(n')}\right| < \frac{\varepsilon}{2}. \tag{3.8}$$

Choose a natural $K \geq \max\left\{N, M, \frac{2}{\varepsilon}\right\}$. Then for all $k > k' \geq K$,

$$\begin{aligned} |b_k - b_{k'}| &= \left|a_{\ell_k}^{(k)} - a_{\ell_{k'}}^{(k')}\right| \\ &\leq \left|a_{\ell_k}^{(k)} - a_{\ell_k}^{(k')}\right| + \left|a_{\ell_k}^{(k')} - a_{\ell_{k'}}^{(k')}\right|. \end{aligned}$$

Since $\ell_k \geq k > K \geq M$ and $k, k' \geq K \geq N$, it follows by (3.8) that

$$\left|a_{\ell_k}^{(k)} - a_{\ell_k}^{(k')}\right| < \frac{\varepsilon}{2}.$$

As (ℓ_n) is strictly increasing, $\ell_k > \ell_{k'}$. It follows by (3.7) that

$$\left|a_{\ell_k}^{(k')} - a_{\ell_{k'}}^{(k')}\right| < \frac{1}{k'} \leq \frac{1}{K} \leq \frac{\varepsilon}{2}.$$

Hence,

$$|b_k - b_{k'}| < \varepsilon,$$

for all $k > k' \geq K$. So $(b_n)_n$ is a Cauchy sequence.

Now consider the sequence $(|\mathbf{a}_n - \mathbf{b}|)_n$. Fix $n \in \mathbb{N}$. Then

$$|\mathbf{a}_n - \mathbf{b}| = \left[\left(\left|a_m^{(n)} - a_{\ell_n}^{(n)}\right|\right)_m\right].$$

Since for all $m \geq \ell_n$,

$$\left|a_m^{(n)} - a_{\ell_n}^{(n)}\right| < \frac{1}{n} = \frac{2}{n} - \frac{1}{n},$$

we obtain

$$|\mathbf{a}_n - \mathbf{b}| = \left[\left(\left|a_m^{(n)} - a_{\ell_n}^{(n)}\right|\right)_m\right] <_{\mathbb{R}} \frac{\mathbf{2}}{\mathbf{n}}.$$

Letting $n \to \infty$, we obtain that $(|\mathbf{a}_n - \mathbf{b}|)$ is a zero-sequence in $\mathbb{R}$. Hence,

$$\lim_{n\to\infty} \mathbf{a}_n = \mathbf{b}.$$

q. e. d.

4 A non-standard approach

Generally a non-standard extension of a linearly ordered algebraic structure A is a new algebraic structure ${}^*A$ which satisfies the same axioms as A but also contains an "infinitely large" element. If we call ${}^\sigma A$ the natural embedding of A in ${}^*A$, this means that ${}^*A$ contains an element ω being greater than any element in ${}^\sigma A$. For semirings, rings and fields, this existence of an infinite element does not contradict any of the other axioms. Therefore a non-standard model exists. As the non-standard model satisfies the same axioms as the standard one, a non-standard ring also contains $-\omega$, which is smaller than any standard element, and a field contains ω^{-1}, which is closer to zero than any non-zero standard element.

There are different ways of constructing non-standard sets. One way is based on *Internal Set Theory*, which is introduced by Nelson in [11]. Internal Set Theory starts with the axioms of Zermelo–Fraenkel set theory with the Axiom of Choice (ZFC) plus a new (undefined) unary predicate "standard" and three additional axiom schemes — *transfer principle, principle of idealisation, principle of standardization.* The transfer priciple states that every statement in first-order logic which holds for a standard set also holds for its non-standard model.

However, Internal Set Theory is more abstract, applicable more generally and thus more powerful than what we need for our purposes. We will concentrate on the construction of the set of hyperrational and later hyperreal numbers using the means which we have established so far, plus a new set theoretical construct called *ultrafilter.* This construction is also called an *ultrapower construction.* It is mainly inspired by Stroyan and Luxemburg [16]. Different approaches are contrasted in further depth in Laugwitz [10].

4.1 Ultrafilters

Initially, we need to introduce the notion of a *free ultrafilter* on an infinite set. We will firstly do this axiomatically and then demonstrate a non-constructive proof for the existence of a free ultrafilter. We use the axioms and terminology for free filters as used in both Zelenyuk [17] and [16].

Definition 4.1. Let J be an infinite set. A set $\mathcal{U} \subseteq \mathcal{P}(J)$ is called a *(proper) filter* on J provided:

$$J \in \mathcal{U} \text{ and } \emptyset \notin \mathcal{U} \quad \textit{(properness)}, \tag{4.1}$$

$$\text{if } A, B \in \mathcal{U}, \text{ then } A \cap B \in \mathcal{U} \quad \textit{(finite intersection property)}, \tag{4.2}$$

$$\text{if } A \in \mathcal{U} \text{ and } A \subseteq B \subseteq J, \text{ then } B \in \mathcal{U} \quad \textit{(superset property)}. \tag{4.3}$$

A filter is called an *ultrafilter* if it is a *maximal filter*, that is, it is not properly contained in any other filter on J.

It is called *free* (or *non-principal*) provided:

$$\text{If } A \in \mathcal{U}, \text{ then } A \text{ is infinite.} \tag{4.4}$$

In the following we will only consider proper filters on an infinite set J.

The first proof of the existence of a free ultrafilter by Tarski dates back to 1930 (see [13] p. 53) and strongly depends on the Axiom of Choice in the form of Zorn's Lemma. We will prove the existence of a free ultrafilter by firstly constructing a free filter and then extending it to a free ultrafilter using the same method as in the proof of the Ultrafilter Theorem ([17] Proposition 2.5).

Definition 4.2. The free filter defined by

$$\mathrm{Fr}(J) := \{A \in \mathcal{P}(J) \mid (J \setminus A) \text{ is finite}\}$$

is called the *Fréchet filter* on J.

Showing that $\mathrm{Fr}(J)$ is indeed a free filter is an easy exercise.

Lemma 4.3. *Every filter containing the Fréchet filter is free.*

Proof. Let $\mathcal{B}$ be an ultrafilter such that $\mathrm{Fr}(J) \subseteq \mathcal{B}$. If $\mathcal{B}$ contained a finite set D, then since $J \setminus D \in \mathrm{Fr}(J) \subseteq \mathcal{B}$, we would obtain $\mathcal{B} \ni (J \setminus D) \cap D = \emptyset$, contradicting properness. Hence, $\mathcal{B}$ is free. **q. e. d.**

Theorem 4.4 (*Ultrafilter Theorem*)**.** *Assuming the Axiom of Choice,* $\mathrm{Fr}(J)$ *can be extended to a free ultrafilter.*

Proof. Let

$$\mathcal{A} := \{\mathcal{F} \subseteq \mathcal{P}(J) \mid \mathrm{Fr}(J) \subseteq \mathcal{F} \text{ and } \mathcal{F} \text{ is a filter}\}\,.$$

Let $\mathcal{B} \subseteq \mathcal{A}$ be any chain, and consider $\bigcup \mathcal{B}$.

Clearly, $\mathrm{Fr}(J) \subseteq \bigcup \mathcal{B}$. So $J \in \mathcal{B}$. Also $\emptyset \notin \bigcup \mathcal{B}$, as the empty set is not contained in any set in $\mathcal{B}$.

Since $\mathcal{B}$ is a chain, $\bigcup \mathcal{B}$ trivially satisfies the superset property.

Let $A, B \in \bigcup \mathcal{B}$. As $\mathcal{B}$ is a chain, there exists a filter $\mathcal{X} \in \mathcal{B}$ such that $A, B \in \mathcal{X}$. Hence, $A \cap B \in \mathcal{X} \subseteq \bigcup \mathcal{B}$. So $\bigcup \mathcal{B}$ has the finite intersection property.

Hence, $\bigcup \mathcal{B}$ is a filter containing $\mathrm{Fr}(J)$, whence $\bigcup \mathcal{B} \in \mathcal{A}$.

By Zorn's Lemma (see Theorem A.7), $\mathcal{A}$ contains a maximal element $\mathcal{M}$, which is thus an ultrafilter.

Finally, as $\mathcal{M}$ contains $\mathrm{Fr}(J)$, it is a free ultrafilter by Lemma 4.3.

q. e. d.

An equivalent condition for maximality is given by the next theorem. We will, however, only present one direction of the implication exploiting Proposition 2.3 from [17].

Theorem 4.5. *An ultrafilter $\mathcal{U}$ has the following property:*

$$\text{If } A \in \mathcal{P}(J), \text{ then either } A \in \mathcal{U} \text{ or } J \setminus A \in \mathcal{U}. \tag{4.5}$$

Proof. Let $A \in \mathcal{P}(J)$.

Case 1: There exists $B \in \mathcal{U}$ such that $B \cap A = \emptyset$. Then $B \subseteq J \setminus A \subseteq J$. Hence, by the superset property, $J \setminus A \in \mathcal{U}$.

Case 2: For every $B \in \mathcal{U}$, we have $B \cap A \neq \emptyset$. Let $\mathcal{F} \in \mathcal{P}(J)$ be defined as the family of non-empty subsets

$$\mathcal{F} := \{C \in \mathcal{P}(J) \mid \exists B \in \mathcal{U} : B \cap A \subseteq C\}.$$

It clearly has the superset property.
For any $C_1, C_2 \in \mathcal{F}$, there exist $B_1, B_2 \in \mathcal{U}$ such that $C_1 \supseteq B_1 \cap A$ and $C_2 \supseteq B_2 \cap A$. Thus,

$$C_1 \cap C_2 \supseteq \underbrace{(B_1 \cap B_2)}_{\in \mathcal{U}} \cap A.$$

Hence, $C_1 \cap C_2 \in \mathcal{F}$. So $\mathcal{F}$ has the finite intersection property.
Consequently, $\mathcal{F}$ is a filter.
Clearly, $\mathcal{U} \subseteq \mathcal{F}$ and $A \in \mathcal{F}$. Since $\mathcal{U}$ is not properly contained in any other filter, it follows $\mathcal{U} = \mathcal{F}$, and thus $A \in \mathcal{U}$.

q. e. d.

Later on, we will refer to property (4.5) as maximality of the ultrafilter.

Remark 4.6. At the point where we use Zorn's Lemma, which is equivalent to the Axiom of Choice, the proof of the existence of a free ultrafilter becomes non-constructive. Hence, we do not obtain any information about any further properties other than the ones which we can directly derive from the axioms it satisfies. As the definition of hyperrationals only depends on the existence of a free ultrafilter, this does not lead to any complications.

4.2 Hyperrationals

First we specify the infinite set J to be the set of natural numbers $\mathbb{N}$. Let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$, and $\mathbb{Q}^{\mathbb{N}}$ the set of rational sequences.

Definition 4.7. We define a relation on $\mathbb{Q}^{\mathbb{N}}$ by

$$(a_n) \sim (b_n) :\Longleftrightarrow \{n \mid a_n = b_n\} \in \mathcal{U}.$$

Proposition 4.8. *The relation $\sim$ defines an equivalence relation.*

Proof. Let $(a_n), (b_n), (c_n) \in \mathbb{Q}^{\mathbb{N}}$.

(Reflexivity) $\{n \mid a_n = a_n\} = \mathbb{N} \in \mathcal{U}$, so $(a_n) \sim (a_n)$.

(Symmetry) Suppose that $(a_n) \sim (b_n)$. Then $\{n \mid b_n = a_n\} = \{n \mid a_n = b_n\} \in \mathcal{U}$. So $(b_n) \sim (a_n)$.

(Transitivity) Suppose that $(a_n) \sim (b_n)$ and $(b_n) \sim (c_n)$. Then

$$\begin{aligned}\{n \mid a_n = c_n\} &\supseteq \{n \mid a_n = c_n \wedge b_n = c_n\} \\ &= \{n \mid a_n = b_n \wedge b_n = c_n\} \\ &= \underbrace{\{n \mid a_n = b_n\}}_{\in\mathcal{U}} \cap \underbrace{\{n \mid b_n = c_n\}}_{\in\mathcal{U}}\end{aligned}$$

Because of the finite intersection and superset property of $\mathcal{U}$, we obtain $\{n \mid a_n = c_n\} \in \mathcal{U}$. So $(a_n) \sim (c_n)$. **q. e. d.**

If $(a_n) \sim (b_n)$, we say that the sequences are $\mathcal{U}$*-equivalent*.

Definition 4.9. The set of *hyperrational numbers* ${}^*\mathbb{Q}$ is defined as the set of all equivalence classes of rational sequences under the equivalence relation $(\sim)$:

$${}^*\mathbb{Q} := \mathbb{Q}^{\mathbb{N}} / \sim .$$

Definition 4.10. Addition, multiplication and an order relation on ${}^*\mathbb{Q}$ are defined as follows:

$$[(a_n)] +_* [(b_n)] := [(a_n + b_n)] \tag{4.6}$$

$$[(a_n)] \cdot_* [(b_n)] := [(a_n \cdot b_n)] \tag{4.7}$$

$$[(a_n)] <_* [(b_n)] :\Longleftrightarrow \{n \mid a_n < b_n\} \in \mathcal{U} \tag{4.8}$$

Proposition 4.11. *Addition and multiplication on* ${}^*\mathbb{Q}$ *are well-defined.*

Proof. First consider addition. Let $(a_n), (a'_n), (b_n), (b'_n) \in \mathbb{Q}^{\mathbb{N}}$ such that $(a_n) \sim (a'_n)$ and $(b_n) \sim (b'_n)$. We need to show that $(a_n + b_n) \sim (a'_n + b'_n)$. Since $\{n \mid a_n = a'_n\} \in \mathcal{U}$ and $\{n \mid b_n = b'_n\} \in \mathcal{U}$, also $\{n \mid a_n = a'_n\} \cap \{n \mid b_n = b'_n\} \in \mathcal{U}$ by the finite intersection property. Hence,

$$\{n \mid a_n + b_n = a'_n + b'_n\} \supseteq \{n \mid a_n = a'_n\} \cap \{n \mid b_n = b'_n\} \in \mathcal{U}.$$

By the superset property, $(a_n + b_n) \sim (a'_n + b'_n)$.

One can give a similar proof for multiplication. **q. e. d.**

Proposition 4.12. *The order relation* $<_*$ *is well-defined.*

Proof. Let $(a_n), (a'_n), (b_n), (b'_n) \in \mathbb{Q}^{\mathbb{N}}$ such that $(a_n) \sim (a'_n)$ and $(b_n) \sim (b'_n)$. Suppose that $\{n \mid a_n < b_n\} \in \mathcal{U}$. Then

$$\begin{aligned}\{n \mid a'_n < b'_n\} &\supseteq \{n \mid a_n < b_n \wedge a_n = a'_n \wedge b_n = b'_n\} \\ &= \underbrace{\{n \mid a_n < b_n\}}_{\in\mathcal{U}} \cap \underbrace{\{n \mid a_n = a'_n\}}_{\in\mathcal{U}} \cap \underbrace{\{n \mid b_n = b'_n\}}_{\in\mathcal{U}} \\ &\in \mathcal{U},\end{aligned}$$

and hence $\{n \mid a'_n < b'_n\} \in \mathcal{U}$ by the finite intersection and superset properties. **q. e. d.**

Remark 4.13. Since we cannot make a statement about the uniqueness of a free ultrafilter $\mathcal{U}$, the set ${}^*\mathbb{Q}$ might not be uniquely defined. We will, however, refer to ${}^*\mathbb{Q}$ as *the* set of hyperrationals, since all of the properties of ultrafilters used to prove properties of ${}^*\mathbb{Q}$ are derived from the axioms for free ultrafilters and will therefore not lead to imprecise statements.

Notation 4.14. We now denote hyperrationals of the form $[(a_n)]$ where (a_n) is a constant sequence $a_n = c$ for some $c \in \mathbb{Q}$ by c_σ. The natural standard copy of $\mathbb{Q}$ in ${}^*\mathbb{Q}$ by this identification is denoted by ${}^\sigma\mathbb{Q}$.

The one additional axiom extending the rational numbers to the hyperrational numbers states the existence of an element $\omega \in {}^*\mathbb{Q}$ which is strictly greater than any standard rational number. We can, for example, define ω as

$$\omega := [(\omega_n)],$$

where $\omega_n = n$.

In order to show that ${}^*\mathbb{Q}$ satisfies all the axioms for a field, we use a trick similar to the one used for our construction of the Cantor real number system, namely, to express ${}^*\mathbb{Q}$ as a quotient ring with its maximal ideal which consists of all elements equivalent to the zero-element of the ring.

First note that $\mathbb{Q}^{\mathbb{N}}$ is a ring. Consider the following subset of $\mathbb{Q}^{\mathbb{N}}$:

$$\left\{ (a_n) \in \mathbb{Q}^{\mathbb{N}} \;\middle|\; (a_n) \sim 0 \right\}.$$

This is infact the set 0_σ, the equivalence class of the constant zero-sequence.

That 0_σ is an ideal follows directly from the finite intersection and superset property of $\mathcal{U}$. So $\mathbb{Q}^{\mathbb{N}}/0_\sigma$ is a ring. Again, we will demonstrate the existence of multiplicative inverses.

Proposition 4.15. *For every $(p_n) \in \mathbb{Q}^{\mathbb{N}}$ such that $(p_n) \not\sim 0_\sigma$ there exists $(q_n) \in \mathbb{Q}^{\mathbb{N}}$ such that $(p_n \cdot q_n) \sim 1_\sigma$.*

Proof. Let $(p_n) \in \mathbb{Q}^{\mathbb{N}}$ such that $(p_n) \not\sim 0_\sigma$. Then $\{n \mid p_n = 0\} \notin \mathcal{U}$. By the maximality of $\mathcal{U}$, $\{n \mid p_n \neq 0\} = \mathbb{N} \setminus \{n \mid p_n = 0\} \in \mathcal{U}$. Let $(q_n) \in \mathbb{Q}^{\mathbb{N}}$ be defined by

$$q_n := \begin{cases} \frac{1}{p_n} & \text{if } p_n \neq 0, \\ 0 & \text{if } p_n = 0. \end{cases}$$

Then $\{n \mid p_n \cdot q_n = 1\} = \{n \mid p_n \neq 0\} \in \mathcal{U}$. Hence, $(p_n \cdot q_n) \sim 1_\sigma$. **q. e. d.**

We denote the multiplicative inverse of $\alpha \in {}^*\mathbb{Q} \setminus \{0_\sigma\}$ by α^{-1}.

Corollary 4.16. $({}^*\mathbb{Q}, +_*, \cdot_*)$ *is a field.*

Proof. Two sequences in $\mathbb{Q}^{\mathbb{N}}$ are $\mathcal{U}$-equivalent if and only if they differ by a sequence which is $\mathcal{U}$-equivalent to the constant zero-sequence. Hence,

$${}^*\mathbb{Q} = (\mathbb{Q}^{\mathbb{N}}/\sim) = \mathbb{Q}^{\mathbb{N}}/0_\sigma.$$

By Proposition 4.15, ${}^*\mathbb{Q}$ is a field. **q. e. d.**

Proposition 4.17. $({}^*\mathbb{Q}, +_*, \cdot_*, <_*)$ *forms an ordered field.*

Proof. Conditions (O1) and (O2) result directly from the order properties of $\mathbb{Q}$.

(O3) Let $(a_n), (b_n), (c_n) \in \mathbb{Q}^{\mathbb{N}}$ such that $[(a_n)] <_* [(b_n)]$ and $[(b_n)] <_* [(c_n)]$. Then

$$\begin{aligned} \{n \mid a_n < c_n\} &\supseteq \{n \mid a_n < b_n \wedge b_n < c_n\} \\ &= \{n \mid a_n < b_n\} \cap \{n \mid b_n < c_n\} \in \mathcal{U}. \end{aligned}$$

Hence, by the superset property, $\{n \mid a_n < c_n\} \in \mathcal{U}$.

(O4) Let $(a_n), (b_n) \in \mathbb{Q}^{\mathbb{N}}$ such that $[(a_n)] \neq [(b_n)]$. Then $\{n \mid a_n = b_n\} \notin \mathcal{U}$. So

$$\{n \mid a_n \neq b_n\} = \mathbb{N} \setminus \{n \mid a_n = b_n\} \in \mathcal{U}.$$

Hence,

$$\begin{aligned} \{n \mid a_n \neq b_n\} &= \{n \mid a_n < b_n \vee a_n > b_n\} \\ &= \{n \mid a_n < b_n\} \mathring{\cup} \{n \mid a_n > b_n\} \in \mathcal{U}. \end{aligned}$$

At least one of the two sets of that union must be infinite. Suppose that $\{n \mid (a_n < b_n)\}$ is infinite. If $\{n \mid (a_n < b_n)\} \in \mathcal{U}$, then $[(a_n)] <_* [(b_n)]$. Otherwise, by maximality and finite intersection property of $\mathcal{U}$,

$$(\mathbb{N} \setminus \{n \mid (a_n < b_n)\}) \cap \{n \mid a_n \neq b_n\} = \{n \mid a_n > b_n\} \in \mathcal{U}.$$

In that case, $[(a_n)] >_* [(b_n)]$.

A similar argument can be applied if $\{n \mid a_n > b_n\}$ is infinite. **q. e. d.**

We have established the ordered field of hyperrationals ${}^*\mathbb{Q}$ with an infinite element ω. Because of the existence of infinitely large and small elements, this field has members which we do not want in the real numbers. In the next section we will show how we can "reduce" the number of elements using, again, equivalence classes.

4.3 Completeness through a quotient ring

Let $\mathcal{O}$ be the ring of *finite* hyperrationals, i. e. the following set:

$$\mathcal{O} := \{\alpha \in {}^*\mathbb{Q} \mid \exists q_\sigma \in {}^\sigma\mathbb{Q} : |\alpha| <_* q_\sigma\}\,.$$

We will continue working with $\mathcal{O}$ so that we do not have to deal with infinitely large hyperrationals.

Next we need to consider hyperrationals which are infinitely close to each other, as we do not want $\mathbb{R}$ to contain such elements. This can be done with an equivalence relation. We will, however, start directly by defining an ideal consisting of the infinitely small hyperrationals (*infinitesimals*) and 0_σ:

$$\mathcal{I} := \{\alpha \in {}^*\mathbb{Q} \mid \forall q_\sigma \in {}^\sigma\mathbb{Q} \setminus \{0_\sigma\} : |\alpha| <_* |q_\sigma|\}\,.$$

Eventually the real numbers will be defined as the quotient ring of $\mathcal{O}$ with $\mathcal{I}$. This quotient ring exists if $\mathcal{O}$ is a ring and $\mathcal{I}$ is an ideal.

Proposition 4.18. *$\mathcal{O}$ is an ordered subring of ${}^*\mathbb{Q}$.*

Proof. To show that $\mathcal{O}$ is a subring of ${}^*\mathbb{Q}$, we only need to check that $\mathcal{O}$ contains 1_σ and is closed under multiplication and subtraction. 1_σ is in ${}^\sigma\mathbb{Q}$ and is thus finite.

Let $\alpha, \beta \in \mathcal{O}$. Then there exist $q_\sigma, p_\sigma \in {}^\sigma\mathbb{Q} \setminus \{0_\sigma\}$ such that $|\alpha| <_* q_\sigma$ and $|\beta| <_* p_\sigma$. Hence,

$$|\alpha \cdot_* \beta| = |\alpha| \cdot_* |\beta| < q_\sigma \cdot_* p_\sigma \in {}^\sigma\mathbb{Q} \setminus \{0_\sigma\}\,,$$

and

$$|\alpha -_* \beta| \leq_* |\alpha| +_* |\beta| <_* q_\sigma + p_\sigma \in {}^\sigma\mathbb{Q} \setminus \{0_\sigma\}\,.$$

Hence, $\alpha \cdot_* \beta$ and $\alpha -_* \beta$ lie in $\mathcal{O}$. **q. e. d.**

Proposition 4.19. *$\mathcal{I}$ is an ideal of $\mathcal{O}$.*

Proof. Let $\alpha, \beta \in \mathcal{I}$ and $\gamma \in \mathcal{O}$, and let $q_\sigma \in {}^\sigma\mathbb{Q} \setminus \{0_\sigma\}$. Then

$$|\alpha| <_* \frac{|q_\sigma|}{2_\sigma},$$

and

$$|\beta| <_* \frac{|q_\sigma|}{2_\sigma}.$$

So

$$|\alpha +_* \beta| \leq_* |\alpha| +_* |\beta| <_* |q_\sigma|.$$

Moreover, there exists $p_\sigma \in {}^\sigma\mathbb{Q}_{>0}$ such that $|\gamma| <_* p_\sigma$.

Since $p_\sigma >_* 0_\sigma$, its inverse p_σ^{-1} exists and lies in ${}^\sigma\mathbb{Q}_{>0}$. So

$$\frac{|q_\sigma|}{p_\sigma} \in {}^\sigma\mathbb{Q} \setminus \{0_\sigma\}.$$

Hence,

$$|\gamma \cdot_* \alpha| = |\gamma| \cdot_* |\alpha| <_* p_\sigma \cdot_* \frac{|q_\sigma|}{p_\sigma} = |q_\sigma|.$$

Thus, $\alpha +_* \beta$ and $\gamma \cdot_* \alpha$ lie in $\mathcal{I}$. **q. e. d.**

Proposition 4.20. *$\mathcal{O}/\mathcal{I}$ forms a field.*

Proof. It is enough to show that every element in $\mathcal{O} \setminus \mathcal{I}$ has a multiplicative inverse in $\mathcal{O}$.

Let $\zeta \in \mathcal{O} \setminus \mathcal{I}$. Then $\zeta \neq 0_\sigma$. Suppose that $\zeta >_* 0_\sigma$. Then ζ^{-1} exists in ${}^*\mathbb{Q}_{>0}$. Moreover, there exist $p_\sigma, q_\sigma \in {}^\sigma\mathbb{Q}_{>0}$ such that

$$p_\sigma <_* \zeta <_* q_\sigma.$$

Hence,

$$p_\sigma^{-1} <_* \zeta^{-1} <_* q_\sigma^{-1},$$

which implies $\zeta^{-1} \in \mathcal{O} \setminus \mathcal{I}$.

A similar argument applies if $\zeta <_* 0_\sigma$. **q. e. d.**

Definition 4.21. Define the field of real numbers $\mathbb{R}$ as the quotient ring

$$\mathbb{R} := \mathcal{O}/\mathcal{I}.$$

Addition and multiplication are induced by the ring $\mathcal{O}$. An order relation on $\mathbb{R}$ is given by

$$\alpha + \mathcal{I} <_\mathbb{R} \beta + \mathcal{I} :\Longleftrightarrow (\alpha <_* \beta \ \wedge \ \alpha + I \neq \beta + I).$$

A proof that $\mathbb{R}$ is ordered can be found in Davis [1] Chapter 2, Theorem 1.8. The embedding of $\mathbb{Q}$ into $\mathbb{R}$ is given by $q \mapsto q_\sigma + \mathcal{I}$. Again, this preserves the properties of $\mathbb{Q}$ as an ordered field, and we can therefore use the same notation on $\mathbb{R}$ as on $\mathbb{Q}$.

Finally, we need to show that $\mathbb{R}$ is complete. We would like to use a notion of completeness which naturally arises from the way we constructed $\mathbb{R}$. This will be discussed in the following section.

4.4 A non-typical notion of completeness

To find a suitable statement of completeness corresponding to the ultrapower construction, we first have to analyse how different versions of the completeness property arise from particular ways of constructing real number systems.

The Dedekind construction of the real numbers first uses Dedekind cuts only on the rational numbers and defines the real numbers as those cuts. Completeness is then the property that every Dedekind cut on the real numbers is represented by exactly one real number, that is, the least upper bound of that particular Dedekind cut. Hence, there is a one-to-one correspondence between Dedekind cuts on the real numbers and the real numbers themselves.

The Cauchy real number system consists of equivalence classes of rational Cauchy sequences. In this context, completeness means that every real Cauchy sequence converges to a unique real number. We obtain a one-to-one correspondence between the equivalence classes of real Cauchy sequences (i. e. sets of Cauchy sequences that tend to the same limit in $\mathbb{R}$) and the real numbers.

In both cases, the idea of completeness is obtained by extending the sets we use on the rationals to construct the reals. Completeness then means that there is a structure-preserving one-to-one correspondence — an order-isomorphism — between the reals and those sets (Dedekind cuts on $\mathbb{R}$ and equivalence classes of real Cauchy sequences).

To apply the same idea to our ultrapower construction, we first need to extend the concept of a non-standard model of the rational numbers to general fields: The corresponding hyperfield ${}^*F$ of an ordered field F is defined to be the set of equivalence classes of sequences in F under the equivalence relation induced by an ultrafilter $\mathcal{U}$. Let $\mathbb{Q}_{^*F}$ be the natural embedding of $\mathbb{Q}$ in ${}^*F$. In analogy to ${}^*\mathbb{Q}$, we define the ring of finite elements in ${}^*F$ as

$$\mathcal{O}_F := \{\alpha \in {}^*F \mid \exists q \in \mathbb{Q}_{^*F} : |\alpha| <_* q\} ,$$

and its ideal of infinitesimals as

$$\mathcal{I}_F := \{\alpha \in {}^*F \mid \forall q \in \mathbb{Q}_{^*F} \setminus \{0_{^*F}\} : |\alpha| <_* |q|\} .$$

Completeness now means that $\mathcal{O}_F/\mathcal{I}_F$ is order-isomorphic to F.

Recalling how ${}^*F$ is constructed by equivalence classes through ultrafilters, one can phrase a sensible statement of completeness:

“An ordered field F is complete if and only if every bounded sequence in F is $\mathcal{U}$-equivalent to a convergent sequence in F, and every convergent sequence in F is $\mathcal{U}$-equivalent to a bounded sequence in F.” (4.9)

For standard fields without infinitely large elements we use the common notion of boundedness and convergence. In this case, the condition that every convergent sequence is $\mathcal{U}$-equivalent to a bounded sequence holds automatically. For a non-standard field F, a sequence (a_n) in F is bounded (by a finite element) if there exists an element q in $\mathbb{Q}_F$, the natural copy of $\mathbb{Q}$ in F, such that for all $n \in \mathbb{N}$,

$$|a_n| <_F q.$$

The sequence converges to a limit $a \in F$ if for all $\varepsilon \in \mathbb{Q}_F$ such that $\varepsilon >_F 0_F$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$|a_n - a| < \varepsilon.$$

(This also gives us a hint as to why non-standard fields are neither Dedekind nor Cauchy complete: A Dedekind cut is not created by a unique element, a Cauchy sequence does not have a unique limit. This is also discussed in Schmieden and Laugwitz [14].)

A proof that for any ordered field F the quotient ring $\mathcal{O}_F/\mathcal{I}_F$ is an ordered field with the supremum property is given in [1] Chapter 2, Theorem 2.5. This verifies that the newly established notion of completeness (4.9) implies supremum-completeness. In fact, as the complete ordered field of real numbers is unique up to isomorphism, this tells us that (4.9) is equivalent to any other version.

The equivalence of different notions of completeness as well as the uniqueness of the real numbers will be discussed in the final chapter.

5 Contrasting methods

5.1 Equivalence of real number systems

In this section we want to show that the Dedekind real number system $\mathbb{R}_\mathrm{D}$ and the Cauchy real number system $\mathbb{R}_\mathrm{C}$ are equivalent and outline a proof that the real numbers are unique up to isomorphism.

We have already shown that $\mathbb{R}_\mathrm{D}$ has the supremum property, only using the properties of Dedekind complete ordered fields (see Theorem 3.16). The converse also holds.

Theorem 5.1. *Every ordered field F which is supremum-complete is also Dedekind complete.*

Proof. Let A be a Dedekind cut on F. Then A is non-empty and bounded above by any element in $F \setminus A$. By the supremum property, $\sup A$ exists in F. **q. e. d.**

The proof that every ordered field which is supremum-complete is also Cauchy complete, and vice versa, can be found in various Real Analysis textbooks. For example, Forster [6] introduces the real numbers as the ordered field which is Cauchy complete and then shows that it is supremum-complete (see [6] § 9 Theorem 3). In contrast, Hart [8] introduces the real numbers with the supremum property and shows that they are Cauchy complete (see [8] Theorem 2.5.5.).

This yields that the three axioms of completeness – Dedekind, Cauchy and supremum-completeness – are equivalent. As a result, every statement about $\mathbb{R}_{\mathrm{D}}$ which is only derived from the complete ordered field properties can also be derived in $\mathbb{R}_{\mathrm{C}}$ and vice versa. Yet, there is one minor detail which needs to be considered: How do we know that we cannot construct an element in $\mathbb{R}_{\mathrm{C}}$ which has no corresponding element in $\mathbb{R}_{\mathrm{D}}$?

This is a crucial point when we use a constructive way of creating a new set opposed to the axiomatic introduction: Even after showing that all the axioms are satisfied, we need to make sure that the new set does not have properties which cannot be derived from those axioms.

It is possible to prove that all ordered fields which satisfy one of the equivalent versions of the completeness axiom are order-isomorphic. A full proof is given in Spivak [15] Chapter 30. We will only outline the proof based on the explanation in [7], Theorem 2.6.

Theorem 5.2. *Any two (Dedekind) complete ordered fields are order-isomorphic.*[2]

Proof. Let F and K be two complete ordered fields.

First note that the natural copies of $\mathbb{Q}$ in F and K, denoted by $\mathbb{Q}_F$ and $\mathbb{Q}_K$ respectively, are order-isomorphic (with a unique isomorphism mapping 1_F to 1_K). Next, since F is complete, every number in F is the least upper bound of a Dedekind cut on $\mathbb{Q}_F$. The same statement holds for K. We can finally extend the isomorphism to the whole sets by mapping $\sup A_F$ onto $\sup A_K$ for every Dedekind cut A_F on $\mathbb{Q}_F$ with corresponding Dedekind cut A_K on $\mathbb{Q}_K$. **q. e. d.**

This finally allows us to talk about *the* complete ordered field of real numbers. As a final result, regardless of how we construct a complete ordered field F, every property of F is also a property of every other complete ordered field.

[2]The proof uses Dedekind completeness, but we could use any other equivalent notion of completeness and find a similar proof.

5.2 Conclusion

We have an intuitive idea of what completeness means: There are no gaps in the real number line. But even after seeing several different ways of filling the gaps in $\mathbb{Q}$, we can still ask the question whether the real numbers do not have any further gaps which could be filled. When looking at non-standard models, in which infinitely small quantities exist, one could say that those hyperreal numbers fill further "gaps" in the real numbers. It is hard to find a general intuitive notion of completeness which fully describes the abstract concept. However, there is a crucial feature of completeness which we described in Section 4.4: If a concept is applied to a field with gaps in order to complete it, the resulting complete field stays unchanged if the same concept is applied again — *One cannot make a complete field more complete.*

Formally, we can only state that an ordered field is complete if and only if it satisfies *a* completeness axiom.

Another question we can now ask is why we have different concepts of completeness if they all turn out to be equivalent. So far we have only worked with completeness on linearly ordered fields, but if we look at the tools which are necessary to define the Dedekind and the Cauchy real number systems, we see that some properties of linearly ordered fields are not fully exploited for the construction of a complete set or the corresponding completeness axiom.

By using Cauchy sequences, the concept of order is reduced to the concept of distance between two points. In the field of Topology, Cauchy sequences and their convergence can be used in general metric spaces. Indeed a metric space is called complete if and only if every Cauchy sequence converges to a point in that space. As mentioned before, in a metric space limits of sequences are always unique. We therefore do not need the uniqueness condition in the statement of Cauchy completeness.

In contrast, Dedekind cuts vastly rely on the order properties of a field. They are still well-defined if the order is not total, i. e. if there are elements which are not comparable under the order relation. Dedekind cuts can therefore be performed on any partially ordered set.

Other constructions lead to other general applications. Different versions of the completeness axiom can be applied to different mathematical constructs.

In Faltin et al. [5], a paper demonstrating yet another construction, the authors conclude:

"Few mathematical structures have undergone as many revisions or have been presented in as many guises as the real numbers. Every generation re-examines the reals in the light of its values and mathematical objectives." ([5] p. 278)

A Appendix

The appendix only includes basic results and their immediate consequences of the lecture courses Real Analysis [3] and Set Theory [12] as well as standard textbooks for these courses.

Definition A.1. A set S equipped with two binary operations addition $(+)$ and multiplication $(\cdot)$ is called a *ring* if it satisfies the following axioms:

(A1) $\forall x, y, z \in S : (x + y) + z = x + (y + z)$ *(associativity of addition)*

(A2) $\forall x, y \in S : x + y = y + x$ *(commutativity of addition)*

There exists an element $0_S \in S$ such that:

(A3) $\forall x \in S : x + 0 = x$ *(existence of an additive identity)*

(A4) $\forall x \in S\ \exists y \in S : x + y = 0_S$ *(existence of additive inverses)*

(M1) $\forall x, y, z \in S : (x \cdot y) \cdot z = x \cdot (y \cdot z)$ *(associativity of multiplication)*

(M2) $\forall x, y \in S : x \cdot y = y \cdot x$ *(commutativity of multiplication)*

There exists an element $1_S \in S$ such that $1_S \neq 0_S$ and the following axiom holds:

(M3) $\forall x \in S : x \cdot 1_S = x$ *(existence of a multiplicative identity)*

(D) $\forall x, y, z \in S : x \cdot (y + z) = (x \cdot y) + (y \cdot z)$ *(distributivity)*

S is called a *field* if it additionally satisfies the axiom:

(M4) $\forall x \in S \setminus \{0_S\}\ \exists y \in S : x \cdot y = 1_S$ *(existence of multiplicative inverses)*

S is called an *ordered* ring (or ordered field respectively) if there is a binary relation $(<)$ defined on S satisfying:

(O1) $\forall x, y, z \in S : [x < y \Longrightarrow x + z < y + z]$

(O2) $\forall x, y, z \in S : [(x < y \wedge 0 < z) \Longrightarrow z \cdot x < z \cdot y]$

(O3) $\forall x, y, z \in S : [(x < y \wedge y < z) \Longrightarrow x < z]$ *(transitivity of order)*

(O4) $\forall x, y \in S$ exactly one of the following holds: $x < y$, $x = y$, or $y < x$ *(trichotomy)*.

We call S an *ordered semiring* if it satisfies the axioms (A1) – (A3), (M1) – (M3), (D) and (O1) – (O4).

Theorem A.2. (see [9] Theorem 8)
For any natural numbers $m, n, k \in \mathbb{N}$, *if* $m + k = n + k$, *then* $m = n$.

Theorem A.3. (see [4] Theorem 4N)
For any natural numbers $m, n, k \in \mathbb{N}$,

$$m + k < n + k \text{ if and only if } m = n.$$

Theorem A.4. (see [9] Theorem 9)
For any two natural numbers $m, n \in \mathbb{N}$, *exactly one of the following three cases holds:*

1. $m = n$,
2. $\exists k \in \mathbb{N} : m = n + k$,
3. $\exists k \in \mathbb{N} : n = m + k$.

Theorem A.5. *Every rational Cauchy sequence is bounded.*

Theorem A.6. *Suppose that* (a_n) *is a rational Cauchy sequence such that* $a_n \neq 0$ *for all* $n \in \mathbb{N}$. *Then* (a_n^{-1}) *is a rational Cauchy sequence.*

Theorem A.7 (Zorn's Lemma, see [4] Theorem 6M)**.** *A set* $\mathcal{B}$ *is a* chain *if and only if for any* $X, Y \in \mathcal{B}$, *either* $X \subseteq Y$ *or* $Y \subseteq X$.

The following statement is equivalent to the Axiom of Choice:
Suppose that $\mathcal{A}$ *is a set such that for every chain* $\mathcal{B} \subseteq \mathcal{A}$, *we have*

$$\bigcup \mathcal{B} \in \mathcal{A}.$$

Then $\mathcal{A}$ *contains a* maximal element M, *that is,* M *is not a subset of any other set in* $\mathcal{A}$.

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