Constructions of the real numbers

a set theoretical approach

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Introduction

$\mathbb{R}$ is widely introduced axiomatically as the (unique) complete linearly ordered field

- What is a real number in set theory?
- What makes $\mathbb{R}$ a linearly ordered field?
- What does completeness mean?
- What ways are there to fill the gaps in $\mathbb{Q}$?

Advice:

"Please forget everything you have learnt in school; for you have not learnt it."

– Edmund Landau in Grundlagen der Analysis (Foundations of Analysis)

Introduction

Process:

- starting from $\mathbb{N}$ with addition and multiplication
- $\mathbb{Z}$ and $\mathbb{Q}$ obtained from $\mathbb{N}$ by elementary set operations
- different concepts applied to $\mathbb{Q}$ to obtain $\mathbb{R}$

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Tools for the natural numbers

$(\mathbb{N}, +, \cdot)$ forms an ordered semiring:

$$
\begin{aligned}
&(m+n)+k = m+(n+k)\\
&m+n = n+m\\
&n+0 = n\\
&(m\cdot n)\cdot k = m\cdot(n\cdot k)\\
&m\cdot n = n\cdot m\\
&n\cdot 1 = n\\
&m\cdot(n+k) = (m\cdot n)+(m\cdot k)
\end{aligned}
$$

Tools for the natural numbers

If $m < n$, then $m + k < n + k$.

If $m < n$ and $0 < k$, then $k \cdot m < k \cdot n$.

If $m < n$ and $n < k$, then $m < k$.

Exactly one of the following holds: $m < n, m = n$, or $m < n$.

From the naturals to the integers

Axiomatically $\mathbb{Z}$ can be introduced as the smallest ordered ring:

- addition, multiplication and an order relation are defined on $\mathbb{Z}$
- $(\mathbb{Z}, +, \cdot, <)$ forms an ordered ring (CRI)
- for every other ordered ring *R*, there exists an injective ring homomorphism from $\mathbb{Z}$ to *R*

From the naturals to the integers

- crucial property of $\mathbb{Z}$: existence of additive inverses

$\rightarrow$ notion of subtraction

- How can we define subtraction only using the tools for $\mathbb{N}$?

From the naturals to the integers

- idea: consider ordered pairs of natural numbers

$\rightarrow$ the pair $(n, m) \in \mathbb{N} \times \mathbb{N}$ corresponds to the integer $n - m$

- problem: no unique representation of integers
  (e.g. $(0, 1)$ and $(2, 3)$ both correspond to -1)

- solution: define an equivalence relation on $\mathbb{N} \times \mathbb{N}$

From the naturals to the integers

Definition

We define an equivalence relation $(\sim)$ on $\mathbb{N} \times \mathbb{N}$ by

$$(n, m) \sim (k, \ell) :\Longleftrightarrow n + \ell = k + m.$$

Use a more familiar notation for equivalence classes:

equivalence class of (n, m) denoted by $[n - m]$

$[n - m] = [k - \ell] \Leftrightarrow n + \ell = k + m$

From the naturals to the integers

Definition

The set of *integers* $\mathbb{Z}$ is defined as the set of all equivalence classes in $\mathbb{N} \times \mathbb{N}$ under $(\sim)$:

$$\mathbb{Z} := (\mathbb{N} \times \mathbb{N} / \sim) = \{[n - m] \mid n, m \in \mathbb{N} \times \mathbb{N}\}.$$

From the naturals to the integers

Definition

For $n, m, k, \ell \in \mathbb{N}$, we define

$$
\begin{aligned}
[n - m] +_{\mathbb{Z}} [k - \ell] &:= [(n + k) - (m + \ell)], \\
[n - m] \cdot_{\mathbb{Z}} [k - \ell] &:= [(n \cdot k + m \cdot \ell) - (n \cdot \ell + m \cdot k)], \\
[n - m] <_{\mathbb{Z}} [k - \ell] &:\Longleftrightarrow n + \ell < k + m.
\end{aligned}
$$

From the naturals to the integers

Definition

Let $n, m, k, \ell \in \mathbb{N}$.
The *negative* of an integer is defined as

$$-[n - m] := [m - n].$$

We also define a new binary operator $(-_{\mathbb{Z}})$ called *subtraction* on the integers:

$$[n - m] -_{\mathbb{Z}} [k - \ell] := [n - m] +_{\mathbb{Z}} (-[k - \ell]).$$

From the naturals to the integers

$\mathbb{N}$ is not a subset of $\mathbb{Z}$!

Define an embedding of $\mathbb{N}$ into $\mathbb{Z}$:

$i : n \mapsto \mathbf{n} := [n - 0]$

$\mathbb{N}_{\mathbb{Z}}$ denotes the image of $\mathbb{N}$ in $\mathbb{Z}$

From the naturals to the integers

Theorem

$$(\mathbb{Z}, +_{\mathbb{Z}}, \cdot_{\mathbb{Z}}, <_{\mathbb{Z}})$$ *forms an ordered ring.*

$\mathbf{0}$ *and* $\mathbf{1}$ *are the additive and multiplicative identities respectively; the additive inverse of an integer* $\mathbf{x}$ *is* $-\mathbf{x}$.

From the naturals to the integers

The structure of $\mathbb{N}$ as its embedding $\mathbb{N}_{\mathbb{Z}}$ is preseverved in $\mathbb{Z}$.

$\Longrightarrow$ We can use the same notation as on $\mathbb{N}$.

From the integers to the rationals

Axiomatically, $\mathbb{Q}$ can be introduced as the smallest ordered field:

- addition, multiplication and an order relation are defined on $\mathbb{Q}$
- $(\mathbb{Q}, +, \cdot, <)$ forms an ordered field
- for every other ordered field K, there exists an injective field homomorphism from $\mathbb{Q}$ to K

From the integers to the rationals

- crucial property of $\mathbb{Q}$: existence of multiplicative inverses

$\rightarrow$ notion of division

- How can we define division only using the properties of $\mathbb{Z}$?

From the integers to the rationals

- idea: consider ordered pairs of integers

$\rightarrow$ the pair $(n, m) \in \mathbb{Z} \times \mathbb{Z}^+$ corresponds to the rational $\frac{n}{m}$

- problem: no unique representation of rationals (e.g. $(1, 2)$ and $(2, 4)$ both correspond to $\frac{1}{2}$)
- solution: define an equivalence relation on $\mathbb{Z} \times \mathbb{Z}^+$

From the integers to the rationals

Definition

We define an equivalence relation $(\sim)$ on $\mathbb{Z} \times \mathbb{Z}^+$ by

$$(n, m) \sim (k, \ell) :\Longleftrightarrow n \cdot \ell = k \cdot m.$$

Use a more familiar notation for equivalence classes:

The equivalence class of (n, m) is denoted by $\frac{n}{m}$.

$\frac{n}{m} = \frac{k}{\ell} \Leftrightarrow n \cdot \ell = k \cdot m.$

From the integers to the rationals

Definition

The set of *rational numbers* $\mathbb{Q}$ is defined as the set of all equivalence classes in $\mathbb{Z} \times \mathbb{Z}^+$ under $(\sim)$:

$$\mathbb{Q} := (\mathbb{Z} \times \mathbb{Z}^+ / \sim) = \left\{ \frac{n}{m} \mid n \in \mathbb{Z}, m \in \mathbb{Z}^+ \right\}.$$

From the integers to the rationals

Definition

For $n, m \in \mathbb{Z}, k, \ell \in \mathbb{Z}^+$, we define

$$\frac{n}{m} +_{\mathbb{Q}} \frac{k}{\ell} := \frac{(n \cdot \ell) + (k \cdot m)}{m \cdot \ell},$$
$$\frac{n}{m} \cdot_{\mathbb{Q}} \frac{k}{\ell} := \frac{n \cdot k}{m \cdot \ell},$$
$$\frac{n}{m} <_{\mathbb{Q}} \frac{k}{\ell} :\Longleftrightarrow n \cdot \ell < k \cdot m.$$

From the integers to the rationals

Definition

Let $n, k \in \mathbb{Z}$ and $m, \ell \in \mathbb{Z}^+$.
The *negative* of a rational is defined as

$$-\frac{n}{m} := \frac{-n}{m}.$$

As on the integers, subtraction $(-_{\mathbb{Q}})$ is defined as

$$\frac{n}{m} -_{\mathbb{Q}} \frac{k}{\ell} := \frac{n}{m} + \left(-\frac{k}{\ell}\right)$$

From the integers to the rationals

Definition

Suppose further that $n \neq 0$. We define the *(multiplicative) inverse* as

$$\left(\frac{n}{m}\right)^{-1} := \begin{cases} \frac{m}{n} & \text{, if } n > 0 \\ \frac{-m}{-n} & \text{, if } n < 0 \end{cases}.$$

We also define a new binary operator $(/_{\mathbb{Q}})$ called *division* on the rationals:

$$\frac{k}{\ell} /_{\mathbb{Q}} \frac{n}{m} := \frac{k}{\ell} \cdot_{\mathbb{Q}} \left(\frac{n}{m}\right)^{-1}.$$

From the integers to the rationals

$\mathbb{Z}$ is not a subset of $\mathbb{Q}$!

Define an embedding of $\mathbb{Z}$ into $\mathbb{Q}$:

$i : n \mapsto \mathbf{n} := \frac{n}{1}$

$\mathbb{Z}_{\mathbb{Q}}$ denotes the image of $\mathbb{Z}$ in $\mathbb{Q}$

From the integers to the rationals

Theorem

$$(\mathbb{Q}, +_{\mathbb{Q}}, \cdot_{\mathbb{Q}}, <_{\mathbb{Q}})$$ *forms an ordered field.*

$\mathbf{0}$ *and* $\mathbf{1}$ *are the additive and multiplicative identities respecitvely; the additive and multiplicative inverses of a rational* $\mathbf{x}$ *are* $-\mathbf{x}$ *and* $\mathbf{x}^{-1}$ *respectively.*

From the integers to the rationals

The structure of $\mathbb{Z}$ as its embedding $\mathbb{Z}_{\mathbb{Q}}$ is preseverved in $\mathbb{Q}$.

$\Longrightarrow$ We can use the same notation as on $\mathbb{Z}$.

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Dedekind's construction through cuts

A Dedekind cut on $\mathbb{Q}$ is a subset of $\mathbb{Q}$ satisfying the following properties:

1. It contains a rational number, but it does not contain all rational numbers.
2. Every rational number of the set is smaller than every rational number not contained in the set.
3. It does not contain a greatest rational number.

Dedekind's construction through cuts

We can formalise this as follows:

Definition (*Dedekind cuts*)

A set $\mathbf{r} \subset \mathbb{Q}$ is called a *(Dedekind) cut* (on $\mathbb{Q}$) if and only if it satisfies the following three conditions:

$$\emptyset \neq \mathbf{r} \neq \mathbb{Q}$$
$$\forall p \in \mathbf{r}\ \forall q \in \mathbb{Q} \setminus \mathbf{r} :\ p < q$$
$$\forall p \in \mathbf{r}\ \exists q \in \mathbf{r} :\ p < q$$

Dedekind's construction through cuts

Definition

The set of real numbers $\mathbb{R}$ is defined as the set of all Dedekind cuts on $\mathbb{Q}$.

Dedekind's construction through cuts

Check list:

- define addition, multiplication and an order relation
- define additive and multiplicative inverses
- define a structure preserving embedding of $\mathbb{Q}$ into $\mathbb{R}$
- check that $\mathbb{R}$ is an ordered field
- check that $\mathbb{R}$ is complete
  – But what does that mean?

Dedekind's construction through cuts

Structure preserving embedding:

$$i : \mathbb{Q} \to \mathbb{R}, \quad q \mapsto \mathbf{q} = \{p \in \mathbb{Q} \mid p < q\}\,.$$

Natural copy of $\mathbb{Q}$ in $\mathbb{R}$:

$$\mathbb{Q}_{\mathbb{R}} := \{\mathbf{q} \mid q \in \mathbb{Q}\}$$

Dedekind's construction through cuts

Order relation on $\mathbb{R}$:

$$\mathbf{r} <_{\mathbb{R}} \mathbf{s} :\Longleftrightarrow \mathbf{r} \subsetneq \mathbf{s}$$

Addition on $\mathbb{R}$:

$$\mathbf{r} +_{\mathbb{R}} \mathbf{s} := \{p + q \mid p \in \mathbf{r}, q \in \mathbf{s}\}$$

Dedekind's construction through cuts

Negative of a real number $\mathbf{r}$:

$$-\mathbf{r} := \{q \in \mathbb{Q} \mid \exists p > q : -p \in \mathbb{Q} \setminus \mathbf{r}\}\,.$$

Modulus:

$$|\mathbf{r}| := \mathbf{r} \cup (-\mathbf{r})$$

Dedekind's construction through cuts

Multiplication on $\mathbb{R}$ for $\mathbf{r}, \mathbf{s} \geq_{\mathbb{R}} \mathbf{0}$:

$$\mathbf{r} \cdot_{\mathbb{R}} \mathbf{s} := \{p \cdot q \mid p \in \mathbf{r} \setminus \mathbf{0}, q \in \mathbf{s} \setminus \mathbf{0}\} \cup \mathbf{0}.$$

In the remaining cases:

$$\mathbf{r} \cdot_{\mathbb{R}} \mathbf{s} := \begin{cases} -(\mathbf{r} \cdot_{\mathbb{R}} |\mathbf{s}|), & \text{if } \mathbf{r} \geq_{\mathbb{R}} \mathbf{0}, \mathbf{s} <_{\mathbb{R}} \mathbf{0} \\ -(|\mathbf{r}| \cdot_{\mathbb{R}} \mathbf{s}), & \text{if } \mathbf{r} <_{\mathbb{R}} \mathbf{0}, \mathbf{s} \geq_{\mathbb{R}} \mathbf{0} \\ |\mathbf{r}| \cdot_{\mathbb{R}} |\mathbf{s}|, & \text{if } \mathbf{r}, \mathbf{s} <_{\mathbb{R}} \mathbf{0}. \end{cases}$$

Dedekind's construction through cuts

Multiplicative inverse:

If $\mathbf{s} >_{\mathbb{R}} \mathbf{0}$, then

$$\mathbf{s}^{-1} := \left\{ q^{-1} \in \mathbb{Q} \mid \exists p \in \mathbb{Q} \setminus \mathbf{s} : p < q \right\} \cup \mathbf{0}.$$

If $\mathbf{s} <_{\mathbb{R}} \mathbf{0}$, then

$$\mathbf{s}^{-1} := -|\mathbf{s}|^{-1}.$$

Dedekind's construction through cuts

Theorem

$(\mathbb{R}, +_{\mathbb{R}}, \cdot_{\mathbb{R}}, <_{\mathbb{R}})$ *forms an ordered field.*

$\mathbf{0}$ *and* $\mathbf{1}$ *are the additive and multiplicative identities respectively; the additive and multiplicative inverses of a real* $\mathbf{x}$ *are* $-\mathbf{x}$ *and* $\mathbf{x}^{-1}$ *respectively.*

Dedekind's construction through cuts

Definition (*Dedekind completeness*)

A complete ordered field F is Dedekind complete if and only if *every Dedekind cut on F has a least upper bound in F.*

Theorem

The Dedekind real number system is Dedekind complete.

Dedekind's construction through cuts

Outline of proof:

Let $\mathbf{A}$ be a Dedekind cut on $\mathbb{R}$

$\mathbf{a} := \bigcup \mathbf{A}$

- show that $\mathbf{a}$ is a Dedekind cut on $\mathbb{Q}$
- show that $\mathbf{b} \leq_{\mathbb{R}} \mathbf{a}$ for all $\mathbf{b} \in \mathbf{A}$
- show that $\mathbf{a} \leq_{\mathbb{R}} \mathbf{c}$ for every upper bound $\mathbf{c}$ of $\mathbf{A}$

Dedekind's construction through cuts

Definition

An ordered field F is supremum complete if and only if every non-empty subset of F which is bounded above has a least upper bound in F.

Dedekind completeness and supremum completeness are equivalent.

Cantor's construction through Cauchy sequences

Idea:

Every real number is the limit point of a rational Cauchy sequence.

Once we have established the real numbers, we want that every real Cauchy sequence converges to a real number.

Cantor's construction through Cauchy sequences

Let C be the set of all Cauchy sequences in $\mathbb{Q}$.
Note that C is a ring.

When do two rational Cauchy sequences represent the same real number?

$\rightarrow$ Whenever they converge to the same limit.

Equivalently without mentioning limits: Whenever they only differ by a zero sequence.

Cantor's construction through Cauchy sequences

Let I be the subset of C of all zero sequences.

$$I := \left\{ (a_n) \in C \mid \lim_{n \to \infty} a_n = 0 \right\}.$$

Note: (a_n) and (b_n) in C represent the same real number if and only if $(a_n) - (b_n) \in I$.

Cantor's construction through Cauchy sequences

Theorem

I is a maximal ideal in C.

Corollary

The quotient ring C/I is a field.

Cantor's construction through Cauchy sequences

Definition

The Cantor real number system $\mathbb{R}$ is defined as the quotient of C with its maximal ideal I:

$$\mathbb{R} := C/I.$$

Addition, multiplication, additive identity and inverses, and multiplicative identity are induced by C.

The natural embedding of $\mathbb{Q}$ into $\mathbb{R}$ is given by constant sequences.

Cantor's construction through Cauchy sequences

Order on $\mathbb{R}$:

$$[(a_n)] <_{\mathbb{R}} [(b_n)] :\iff (\exists \delta \in \mathbb{Q}_{>0}\ \exists N \in \mathbb{N}\ \forall n \geq N : a_n > b_n + \delta).$$

Cantor's construction through Cauchy sequences

Theorem

$(\mathbb{R}, +_{\mathbb{R}}, \cdot_{\mathbb{R}}, <_{\mathbb{R}})$ *forms an ordered field.*

Cantor's construction through Cauchy sequences

Definition (*Cauchy completeness*)

A complete ordered field F is Cauchy complete if and only if *every Cauchy sequence in F has a limit point in F.*

Theorem

The Cantor real number system is Cauchy complete.

Cantor's construction through Cauchy sequences

Outline of proof:

Let $(\mathbf{a}_n)_n$ be a Cauchy sequence in $\mathbb{R}$.

Every member of the sequence is of the form $\mathbf{a}_n = \left(a_m^{(n)}\right)_m + I$, where $\left(a_m^{(n)}\right)_m$ is a rational Cauchy sequence.

Choose suitable $\ell_n \in \mathbb{N}$ such that for all $m, m' \geq \ell_n$, $\left|a_m^{(n)} - a_{m'}^{(n)}\right| < \frac{1}{n}$.

- show that $\mathbf{b} = (b_n)_n := \left(a_{\ell_n}^{(n)}\right)_n$ is a rational Cauchy sequence
- show that $(\mathbf{a}_n)_n$ converges to $\mathbf{b}$

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Non-standard models

Non-standard model of an algebraic construct: satisfies the usual axioms and contains an infinitely large element.

Different approaches to non-standard models of algebraic constructs:

- Internal Set Theory: Extend ZFC by further axioms to create non-standard sets
- Algebraic Extension: Adjoin an infinite element to the field, ring, semiring etc.
- Ultrapower construction: Use sequences on ultrafilters.

Ultrafilters

A set $\mathcal{U} \in \mathcal{P}(\mathbb{N})$ is called a (proper) filter on $\mathbb{N}$ provided:

- $\mathbb{N} \in \mathcal{U}$ and $\emptyset \notin \mathcal{U}$ (properness)
- if $A, B \in \mathcal{U}$, then $A \cap B \in \mathcal{U}$ (finite intersection property)
- if $A \in \mathcal{U}$ and $A \subseteq B \subseteq \mathbb{N}$, then $B \in \mathcal{U}$ (superset property)

Ultrafilters

A filter is called ultrafilter if it is a maximal filter:

$$\text{If } A \in \mathcal{P}(\mathbb{N}), \text{ then either } A \in U \text{ or } \mathbb{N} \setminus A \in \mathcal{U}.$$

It is called free if it has the freeness property:

$$\text{If } A \in \mathcal{U}, \text{ then } A \text{ is infinite.}$$

Ultrafilters

- a free ultrafilter exists (Tarski 1930)
- the proof requires the Axiom of Choice in the form of Zorn's Lemma
- it is non-constructive!

Hyperrationals

Let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$.
Consider the ring of rational sequences $\mathbb{Q}^{\mathbb{N}}$.

- Define an equivalence relation on $\mathbb{Q}^{\mathbb{N}}$:

$$(a_n) \sim (b_n) :\Longleftrightarrow \{n \in \mathbb{N} \mid a_n = b_n\} \in \mathcal{U}$$

- Two sequences are equivalent if and only if they agree on a set in $\mathcal{U}$.

Hyperrationals

Definition

The set of hyperrational numbers ${}^*\mathbb{Q}$ is the set of all equivalence classes in $\mathbb{Q}^{\mathbb{N}}$ under $(\sim)$.

A natural embedding of $\mathbb{Q}$ into ${}^*\mathbb{Q}$ is given by constant sequences. We denote it by ${}^\sigma\mathbb{Q}$ and its elements by q_σ.

Hyperrationals

Definition

Define an order relation ($<_*$) on ${}^*\mathbb{Q}$ by

$$[(a_n)] <_* [(b_n)] :\Longleftrightarrow \{n \in \mathbb{N} \mid a_n < b_n\} \in \mathcal{U}.$$

- define a sequence $\omega_n = n$ in $\mathbb{Q}^{\mathbb{N}}$
- let $\omega = [(\omega_n)] \in {}^*\mathbb{Q}$
- $\forall q_\sigma \in {}^\sigma\mathbb{Q}: \; q_\sigma <_* \omega$

$\Rightarrow$ an infinite element exists

Hyperrationals

- To show: ${}^*\mathbb{Q}$ is a field
- note that $(a_n) \sim (b_n)$ if and only if $(a_n) - (b_n) \sim 0_\sigma$
- Idea: express ${}^*\mathbb{Q}$ as a quotient of $\mathbb{Q}^{\mathbb{N}}$ with a maximal ideal

Hyperrationals

$I := \{(a_n) \in {}^*\mathbb{Q} \mid (a_n) \sim o_n\}$, where (o_n) is the constant zero sequence

Theorem

I is a maximal ideal of $\mathbb{Q}^{\mathbb{N}}$. Hence, $\mathbb{Q}^{\mathbb{N}}/I$ is a field.

Theorem

${}^*\mathbb{Q} = \mathbb{Q}^{\mathbb{N}}/I$, and ${}^*\mathbb{Q}$ forms an ordered field.

Completeness through a quotient ring

- Problem: ${}^*\mathbb{Q}$ contains infinitely large elements.

$\rightarrow$ Consider the ring of finite hyperrationals
$\mathcal{O} := \{a \in {}^*\mathbb{Q} \mid \exists p_\sigma \in {}^\sigma\mathbb{Q} : a <_* p_\sigma\}$

- Problem: $\mathcal{O}$ contains infinitely small elements.

$\rightarrow$ form the quotient with the ideal of infinitely small hyperrationals and zero
$o := \{a \in {}^*\mathbb{Q} \mid \forall p_\sigma \in {}^\sigma\mathbb{Q} : |a| <_* |p_\sigma|\} \cup \{0_\sigma\}$

Completeness through a quotient ring

Theorem

o is a maximal ideal of $\mathcal{O}$. Hence, $\mathcal{O}/o$ is a field.

Definition

Define the set of real numbers obtained by a ultrapower construction by

$$\mathbb{R} := \mathcal{O}/o.$$

A non-typical notion of completeness

- What is the naturally arising version of completeness?
- Look at the standard approaches to completeness:
  - Dedekind cuts on $\mathbb{Q}$ correspond to real numbers
  - $\rightarrow$ completeness: there is a one-one correspondence between Dedekind cuts on $\mathbb{R}$ and real numbers
  - equivalence classes of Cauchy sequences on $\mathbb{Q}$ correpsond to real numbers
  - $\rightarrow$ completeness: every Cauchy sequence in $\mathbb{R}$ corresponds to a real number

A non-typical notion of completeness

General process:

- find a concept to fill the gaps in $\mathbb{Q}$ to obtain $\mathbb{R}$
- completeness: the same concept applied to $\mathbb{R}$ results in $\mathbb{R}$

$\Longrightarrow$ *"One cannot make a complete field completer."*

A non-typical notion of completeness

Suggestion for a non-typical notion of completeness:
Let F be an ordered field.

> F is complete if and only if every bounded sequence in F is $\mathcal{U}$-equivalent to a convergent sequence in F.

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Equivalence of real number systems

- all notions of completeness (Dedekind, Cantor, supremum completeness) are equivalent:
  a field which is complete w. r. t. one version is also complete w. r. t. the others
- How do we know that the Dedekind real number system does not have properties that the Cantor real number system doesn't, or vice versa?

Equivalence of real number systems

Theorem

All sets satisfying the axioms of a complete ordered field are order-isomorphic.

Outline of proof:

Construct an order-isomorphism ϕ from F to K as follows:

- ϕ maps the subfield Q_F generated by 1_F to the subfield Q_K generated by 1_K (both isomorphic to $\mathbb{Q}$)
- ϕ maps the least upper bound of a Dedekind cut on Q_F to the least upper bound of the corresponding Dedekind cut on Q_K
- show that ϕ is an order-isomorphism

Conclusion

- The complete ordered field of real numbers $\mathbb{R}$ is unique.

- Why are different notions of completeness useful?

$\rightarrow$ Note that we did not used all the properties of the ordered field $\mathbb{Q}$ to fill the gaps.

Conclusion

- Dedekind cuts can be performed on partially ordered sets.
- Cauchy sequences only require a notion of distance, not of order. (General applications in Topology.)

Conclusion

"Few mathematical structures have undergone as many revisions or have been presented in as many guises as the real numbers. Every generation re-examines the reals in the light of its values and mathematical objectives."

– F. Faltin, N. Metropolis, B. Ross and G.-C. Rota in *The Real Numbers as a Wreath Product*